Identification of Structural Vector Autoregressions Through Higher Unconditional Moments*

Alain Guay†

November 12, 2020

Abstract

This file contains the supplementary appendices.

*Correspondence: Alain Guay, Department of Economics, ESG-UQAM Montréal, 3120 Sainte-Catherine est, Montréal, Québec, Canada, H2X 3X2. Tel.: 1-514-987-3000 8377. E-mail: guay.alain@uqam.ca.

†Université du Québec à Montréal and Chaire en macroéconomie et prévisions ESG-UQAM.

Supplemental Material

Appendix A presents the rank condition for the local, statistical identification of SVAR processes with asymmetric structural shocks. Appendix B details the analytical partial derivatives involved in the Jacobian matrices related to the rank condition. Appendix C derives the rank condition. Appendix D contains the derivation of the asymptotic distribution of the rank test and a justification of the bootstrap procedure when $r^* > 0$. Appendix E documents the empirical sizes and powers of rank tests for symmetry. Appendix F reports the estimates of the structural parameters involved in system (30).

Appendix A: Identication under asymmetric structural shocks

The appendix elaborates the rank condition for a case which exploits only the skewness of the structural shocks. For this case, the relation between the reduced-form innovations and the structural shocks is partitioned as:

$$\nu_t = \begin{pmatrix} \Theta_s & \Theta_{ns} \end{pmatrix} \begin{pmatrix} \epsilon_{s,t} \\ \epsilon_{ns,t} \end{pmatrix}, \tag{A.1}$$

where $\epsilon_{s,t}$ and $\epsilon_{ns,t}$ contain the m_s and $(n - m_s)$ asymmetric and symmetric structural shocks.

Here, the number of parameters to identify is $\eta = n^2 + m_s$ because there are n^2 and m_s parameters to identify in Θ and S_ϵ. From the reduced form, $\rho = \left[\frac{n(n+1)}{2}\right] + \left[\frac{n(n+1)(n+2)}{6}\right]$ since there are $\frac{n(n+1)}{2}$ and $\frac{n(n+1)(n+2)}{6}$ distinct elements in Σ_ν and S_ν. The information contained in S_ν contributes to identify the parameters in Θ_s and S_ϵ, whereas Σ_ν contains specific information to identify the parameters in Θ_{ns}.

The sufficient rank condition holds when $r = \eta$. Under the short-run restrictions $R\theta_{ns} = q$, the rank condition is verified if:

$$rk[J^+] = rk\begin{bmatrix} J^+_{\theta_s} & J^+_{\theta_{ns}} & J^+_{s_\epsilon} \end{bmatrix} = rk\begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & J_{\sigma_\nu,s_\epsilon} \\ J_{s_\nu,\theta_s} & J_{s_\nu,\theta_{ns}} & J_{s_\nu,s_\epsilon} \\ 0 & R & 0 \end{bmatrix} = \eta, \tag{A.2}$$

where J^+ is the augmented Jacobian matrix, $J^+_{\theta_s} = \begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} & 0' \end{bmatrix}'$, $J^+_{\theta_{ns}} = \begin{bmatrix} J'_{\sigma_\nu,\theta_{ns}} & J'_{s_\nu,\theta_{ns}} & R' \end{bmatrix}'$, $J^+_{s_\epsilon} = \begin{bmatrix} J'_{\sigma_\nu,s_\epsilon} & J'_{s_\nu,s_\epsilon} & 0' \end{bmatrix}'$, and $J_{y,x} = \frac{\partial y}{\partial x'}$. Moreover, the vector σ_ν vectorizes the lower triangular part of the symmetric covariance matrix Σ_ν, and the vector s_ν collects the distinct elements of the coskewness matrix S_ν. Finally, the vector θ_s stacks the columns of the matrix Θ_s in system (A.1), the vector θ_{ns} contains the elements of the matrix Θ_{ns} and the vector s_ϵ includes the non-zero elements of the skewness matrix S_ϵ.

When no restrictions are placed on the structural parameters ($R = 0$), the rank of J is given by

$$rk[J] = rk\begin{bmatrix} J_{\theta_s} & J_{\theta_{ns}} & J_{s_\epsilon}\end{bmatrix} = rk\begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & J_{\sigma_\nu,s_\epsilon} \\ J_{s_\nu,\theta_s} & J_{s_\nu,\theta_{ns}} & J_{s_\nu,s_\epsilon}\end{bmatrix}$$

which is equal to $r = r_s + r_{ns} + r_{s_\epsilon}$ with $r_s = rk[J_{\theta_s}] = n \times m_s$, $r_{ns} = rk[J_{\theta_{ns}}] = \frac{n(n+1)}{2} - \frac{m_s(m_s+1)}{2}$, and $r_{s_\epsilon} = rk\left[J_{s_\epsilon}\right] = m_s$ as we show below. Consequently, the entire structural system is locally, statistically identified ($\eta = r$) when at least all, but one, structural shocks display non-zero skewnesses. Also, whether or not $\eta = r$, the parameters involve in Θ_s and S_ϵ are locally, statistically identified through the information contained in Σ_ν and S_ν. Hence, if the structural shocks of interest are asymmetric, then their effects are identified. When some restrictions are imposed on the structural parameters ($R \neq 0$), then the entire structural system is locally, statistically identified when ($\eta - r$) linearly independent restrictions are imposed on the structural parameters contained in θ_{ns}. Thus, if the structural shocks of interest are symmetric, then their effects can only be determined when ($\eta - r$) restrictions are placed on Θ_{ns}.

Appendix B: Analytical derivatives involved in the Jacobian matrices

This appendix presents the analytical partial derivatives involved in the Jacobian matrices for the cases (A.2), (17) and (18). First, the partial derivatives of the second unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$\begin{aligned} J_{\sigma_\nu,\theta_i} &= 2D_\sigma^+(\Theta \otimes I_n)\Upsilon_{\theta_i}, \\ J_{\sigma_\nu,s_\epsilon} &= 0, \\ J_{\sigma_\nu,\kappa_\epsilon^e} &= 0, \end{aligned}$$

where $i = s, ns$ in (A.1), $i = \kappa, n\kappa$ in (17), and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (18). The vectorization of the distinct elements of the second moments yields $\sigma_\nu = D_\sigma^+ vec(\Sigma_\nu)$, where $\sigma_\nu = vech(\Sigma_\nu)$, $D_\sigma^+ = (D_\sigma' D_\sigma)^{-1}D_\sigma'$, and D_σ is the $\left(n^2 \times \frac{n(n+1)}{2}\right)$ duplication matrix such that $D_\sigma \sigma_\nu = vec(\Sigma_\nu)$. Using this vectorization, we obtain $\frac{\partial\sigma_\nu}{\partial\theta_i'} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(\Theta)'}\frac{\partial vec(\Theta)}{\partial\theta_i'}$. Equation (11) leads to $vec(\Sigma_\nu) = (\Theta \otimes \Theta)vec(I_n)$, so that $\frac{\partial vec(\Sigma_\nu)}{\partial vec(\Theta)'} = 2(\Theta \otimes I_n)$ (see Lütkepohl, 2007, p. 363). Also, $\frac{\partial vec(\Theta)}{\partial\theta_i'} = \Upsilon_{\theta_i}$ is a matrix containing the values one and zero such that only the partial derivatives with respect to the elements of the vector θ_i are selected. As an example, consider the relation (A.1) with $n = 2$ and $m_s = 1$ (where the asymmetric structural shock is ordered first), then the ($n^2 \times nm_s$) selection matrix corresponds to $\Upsilon_{\theta_s} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0\end{pmatrix}'$ and $\theta_s = vec(\Theta_s)$. Moreover, $\frac{\partial\sigma_\nu}{\partial s_\epsilon'} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(S_\epsilon)'}\frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'}$, where $\frac{\partial vec(\Sigma_\nu)}{\partial vec(S_\epsilon)'} = 0$ given that Σ_ν is not a function of the skewnesses of the structural shocks. Likewise, $\frac{\partial\sigma_\nu}{\partial\kappa_\epsilon^{e\prime}} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(K_\epsilon^e)'}\frac{\partial vec(K_\epsilon^e)}{\partial\kappa_\epsilon^{e\prime}}$ with $\frac{\partial vec(\Sigma_\nu)}{\partial vec(K_\epsilon^e)'} = 0$.

Next, the partial derivatives of the third unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$\begin{aligned}
J_{s_\nu,\theta_i} &= D_s^+\{(I_{n^2}\otimes\Theta S_\epsilon)[(I_n\otimes C_{n,n}\otimes I_n)[(I_{n^2}\otimes vec(\Theta'))+(vec(\Theta')\otimes I_{n^2})]C_{n,n}] \\
&\quad +[(\Theta\otimes\Theta)S_\epsilon'\otimes I_n]\}\Upsilon_{\theta_i}, \\
J_{s_\nu,s_\epsilon} &= D_s^+(\Theta\otimes\Theta\otimes\Theta)\Upsilon_{s_\epsilon}, \\
J_{s_\nu,\kappa_\epsilon^e} &= 0,
\end{aligned}$$

where $i=s,ns$ in (A.1) and $i=ss,\kappa\kappa,s\kappa,ns\kappa$ in (18). The vectorization of the distinct elements of the third moments corresponds to $s_\nu=D_s^+vec(S_\nu)$, where $D_s^+=(D_s'D_s)^{-1}D_s'$, and D_s is the $\left(n^3\times\frac{n(n+1)(n+2)}{6}\right)$ matrix such that $D_s s_\nu=vec(S_\nu)$. As an example, for a bivariate system with $n=2$, then:

$$D_s=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&1&0&0\\0&0&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Using the above vectorization, we have $\frac{\partial s_\nu}{\partial\theta_i'}=D_s^+\frac{\partial vec(S_\nu)}{\partial vec(\Theta)'}\frac{\partial vec(\Theta)}{\partial\theta_i'}$ with $\frac{\partial vec(\Theta)}{\partial\theta_i'}=\Upsilon_{\theta_i}$. Rewriting equation (12) as $vec(S_\nu)=[(\Theta\otimes\Theta)\otimes\Theta]vec(S_\epsilon)$, then $\frac{\partial vec(S_\nu)}{\partial vec(\Theta)'}=(I_{n^2}\otimes\Theta S_\epsilon)\frac{\partial vec(\Theta'\otimes\Theta')}{\partial vec(\Theta)'}+[(\Theta\otimes\Theta)S_\epsilon'\otimes I_n]$, where $\frac{\partial vec(\Theta'\otimes\Theta')}{\partial vec(\Theta)'}=(I_n\otimes C_{n,n}\otimes I_n)[(I_{n^2}\otimes vec(\Theta'))+(vec(\Theta')\otimes I_{n^2})]\frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$ with $\frac{\partial vec(\Theta')}{\partial vec(\Theta)'}=C_{n,n}$ (see Magnus and Neudecker, 2007, pp. 208–209), and $C_{n,m}$ is a $(nm\times nm)$ commutation matrix implying that $C_{n,m}vec(A)=vec(A')$ for the arbitrary $(n\times m)$ matrix A. Note that $\frac{\partial s_\nu}{\partial\theta_i'}=0$ for $i=ns$ in (A.2) and for $i=\kappa\kappa,ns\kappa$ in (18), since S_ν is not a function of the structural parameters relating the reduced-form innovations to the symmetric structural shocks. Furthermore, $\frac{\partial s_\nu}{\partial s_\epsilon}=D_s^+\frac{\partial vec(S_\nu)}{\partial vec(S_\epsilon)'}\frac{\partial vec(S_\epsilon)}{\partial s_\epsilon{}'}$, where $\frac{\partial vec(S_\nu)}{\partial vec(S_\epsilon)'}=(\Theta\otimes\Theta\otimes\Theta)$ and $\frac{\partial vec(S_\epsilon)}{\partial s_\epsilon{}'}=\Upsilon_{s_\epsilon}$ is a $(n^3\times m_s)$ matrix selecting the partial derivatives with respect to the non-zero elements of s_ϵ. In particular, for a system with $n=m_s=2$, then Υ_{s_ϵ} has values one for the (1,1) and (8,2) elements, and zero elsewhere. For the system with $n=2$ and $m_s=1$, then Υ_{s_ϵ} has values one for the (1,1) element, and zero elsewhere. Moreover, $\frac{\partial s_\nu}{\partial\kappa_\epsilon^e}=D_s^+\frac{\partial vec(S_\nu)}{\partial vec(K_\epsilon^e)'}\frac{\partial vec(K_\epsilon^e)}{\partial\kappa_\epsilon^e{}'}$, where $\frac{\partial vec(S_\nu)}{\partial vec(K_\epsilon^e)'}=0$ given that S_ν is not a function of the excess kurtoses of the structural shocks.

Let us now examine the rank of the matrices J_{σ_ν,θ_s}, $J_{s_\nu,\theta_{ns}}$ and J_{s_ν,s_ϵ}. As illustration, consider a system with $n=2$,

$$\begin{pmatrix}\nu_{1,t}\\\nu_{2,t}\end{pmatrix}=\begin{pmatrix}\theta_{11}&\theta_{12}\\\theta_{21}&\theta_{22}\end{pmatrix}\begin{pmatrix}\epsilon_{1,t}\\\epsilon_{2,t}\end{pmatrix} \tag{A.1}$$

For this example, the Jacobian matrix of the derivatives of the covariance matrix with respect to the parameters Θ is given by

$$J_{\sigma_\nu,\theta} = \begin{bmatrix} 2\theta_{11} & 0 & : & 2\theta_{12} & 0 \\ \theta_{21} & \theta_{11} & : & \theta_{22} & \theta_{12} \\ 0 & 2\theta_{21} & : & 0 & 2\theta_{22} \end{bmatrix}.$$

where $\theta = vec(\Theta)$. For a full rank matrix Θ, this matrix $J_{\sigma_\nu,\theta}$ is of rank $\frac{n(n+1)}{2}$ and each $\frac{n(n+1)}{2} \times n$ submatrix corresponding to the derivatives of $J_{\sigma_\nu,\theta}$ with respect to a column of the matrix Θ is of rank equals to n and this holds for $\forall\ n$. Also, the Jacobian matrix of the coskewness $J_{s_\nu,\theta}$ with respect to Θ for (A.1) is

$$J_{s_\nu,\theta} = \begin{bmatrix} 3\theta_{11}^2 s_{\epsilon,1,11} & 0 & : & 3\theta_{12}^2 s_{\epsilon,2,22} & 0 \\ 2\theta_{11}\theta_{21}s_{\epsilon,1,11} & \theta_{11}^2 s_{\epsilon,1,11} & : & 2\theta_{12}\theta_{22}s_{\epsilon,2,22} & \theta_{12}^2 s_{\epsilon,2,22} \\ \theta_{21}^2 s_{\epsilon,1,11} & 2\theta_{21}\theta_{11}s_{\epsilon,1,11} & : & \theta_{22}^2 s_{\epsilon,2,22} & 2\theta_{12}\theta_{22}^2 s_{\epsilon,2,22} \\ 0 & 3\theta_{21}^2 s_{\epsilon,1,11} & : & 0 & 3\theta_{22}^2 s_{\epsilon,2,22} \end{bmatrix}.$$

For a full rank matrix Θ, the Jacobian matrix $J_{s_\nu,\theta}$ of dimension $\frac{n(n+1)(n+2)}{6} \times n^2$ is of rank $n \times m_s$ which equals the rank of the matrix J_{s_ν,θ_s} since $J_{s_\nu,\theta_s} = J_{s_\nu,\theta}\Upsilon_{\theta_s}$. In the case above, for $m_s = 1$ (for instance when $s_{\epsilon,1,11} \neq 0$ and $s_{\epsilon,2,22} = 0$), the matrix J_{s_ν,θ_s} corresponds to the first two columns of $J_{s_\nu,\theta}$, whereas $J_{s_\nu,\theta_{ns}}$ corresponds to the two last columns of $J_{s_\nu,\theta}$. The rank of J_{s_ν,θ_s} and $J_{s_\nu,\theta}$ is equal to $n \times m_s = 2$. For $m_s = 2$, $J_{s_\nu,\theta_s} = J_{s_\nu,\theta}$ and the rank is $n \times m_s = 4$. For the general case, rearranging the rows of the matrix $J_{s_\nu,\theta}$ corresponding to the k-th column vector $\theta_{\bullet,k}$ of the matrix Θ, leads to the following $\frac{n(n+1)(n+2)}{6} \times n$ matrix

$$J^*_{s_\nu,\theta_{\bullet,k}} = \begin{bmatrix} B_{1k} \\ B_{2k} \\ \dots \\ B_{nk} \\ C_k \end{bmatrix} s_{\epsilon,k,kk}$$

where the matrix C_i is of dimension $\left(\frac{n(n+1)(n+2)}{6} - n^2\right) \times n$ for $n > 2$. The $n \times n$ matrices B_{lk} are given by $B_{lk} = \frac{\partial s_{\nu,l,l,j}}{\partial \theta'_{\bullet,k}}$ for $k, l, j = 1, \dots, n$ and C_k contains the derivatives of $s_{\nu,i,j,l}$ respective to $\theta'_{\bullet,k}$ for all $i < j < l$ for $i, j, l = 1, \dots, n$. Note that the column rank of $J_{s_\nu,\theta_{\bullet,k}}$ is the same as $J^*_{s_\nu,\theta_{\bullet,k}}$. Each matrix B_{lk} has the term $\theta_{lk}^2 s_{\epsilon,k,kk}$ on its diagonal except at the element l, k which is $3\theta_{lk}^2 s_{\epsilon,k,kk}$. The matrices B_{lk} are then of full column rank for all $\theta_{lk} \neq 0$. Given that Θ is of full rank, $J^*_{s_\nu,\theta_{\bullet,k}}$ (and then $J_{s_\nu,\theta_{\bullet,k}}$) is necessarily of full rank for $s_{\epsilon,k,kk} \neq 0$ and $J_{s_\nu,\theta_{\bullet,k}}$ cannot be collinear with $J_{s_\nu,\theta_{\bullet,k}}$ for $k \neq l$, $s_{\epsilon,k,kk} \neq 0$ and $s_{\epsilon,l,ll} \neq 0$. This shows that the Jacobian matrix $J_{s_\nu,\theta}$ is of rank equals to $n \times m_s$. For the illustration with $n = 2$, we get

$$J_{s_\nu,\theta} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

where B_{lk} are 2×2 matrices for $k, l = 1, 2$. For this case, $\frac{n(n+1)(n+2)}{6} - n^2 = 0$, so that there is no matrix C_k. We see that for each submatrix B_{lk}, the diagonal elements are function of $\theta_{lk}^2 s_{\epsilon,k,kk}$. For a full rank matrix Θ, the first two columns corresponding to B_{11} and B_{21} are of full rank (when $s_{\epsilon,1,11} \neq 0$) and they cannot be colinear with the last two columns corresponding to B_{12} and B_{22} (when $s_{\epsilon,2,22} \neq 0$).

For the Jacobian matrix J_{s_ν,s_ϵ}, the rank can be easily shown. The expression $(\Theta \otimes \Theta \otimes \Theta)$ is a square full rank matrix, so $(\Theta \otimes \Theta \otimes \Theta)\Upsilon_{s_\epsilon}$ is of the same column rank than Υ_{s_ϵ}, namely m_s. Since D_s^+ is a full column rank, $D_s^+(\Theta \otimes \Theta \otimes \Theta)\Upsilon_{s_\epsilon}$ has a rank equals to m_s.[1] For (A.1),

$$J_{s_\nu,s_\epsilon} = \begin{bmatrix} \theta_{11}^3 & : & \theta_{12}^3 \\ \theta_{11}^2\theta_{21} & : & \theta_{12}^2\theta_{22} \\ \theta_{11}\theta_{21}^2 & : & \theta_{12}\theta_{22}^2 \\ \theta_{21}^3 & : & \theta_{22}^3 \end{bmatrix} \Upsilon_{s_\epsilon}.$$

The rank of this matrix equals the rank of Υ_{s_ϵ} which equals m_s. However, the rank of $[J_{s_\nu,\theta} \;\; J_{s_\nu,s_\epsilon}]$ equals the rank of the matrix $J_{s_\nu,\theta}$ namely $n \times m_s$ given that $J_{s_\nu,\theta_{\bullet,k}} \times \theta_{\bullet,k} = 3 s_{\epsilon,k,kk} J_{s_\nu,s_\epsilon,k}$ where k indexes the column of the respective matrix. This holds for $\forall\, n$ for a full rank matrix Θ.

Finally, the partial derivatives of the fourth unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$\begin{aligned}
J_{\kappa_\nu^e,\theta_i} &= D_\kappa^+\{(I_{n^2} \otimes \Theta K_\epsilon^e)(I_{n^2} \otimes C_{n,n^2} \otimes I_n)[(I_{n^4} \otimes vec(\Theta'))(I_n \otimes C_{n,n} \otimes I_n) \times [(I_{n^2} \otimes vec(\Theta') \\
&\quad +(vec(\Theta') \otimes I_{n^2})]C_{n,n} + (vec(\Theta' \otimes \Theta') \otimes I_{n^2})C_{n,n}] + [(\Theta \otimes \Theta \otimes \Theta)K_\epsilon^{e\prime} \otimes I_n]\}\Upsilon_{\theta_i}, \\
J_{\kappa_\nu^e,s_\epsilon} &= 0, \\
J_{\kappa_\nu^e,\kappa_\epsilon^e} &= D_\kappa^+(\Theta \otimes \Theta \otimes \Theta \otimes \Theta)\Upsilon_{\kappa_\epsilon^e},
\end{aligned}$$

where $i = \kappa, n\kappa$ in (17) and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (18). The vectorization of the distinct elements of the fourth moments is $\kappa_\nu^e = D_\kappa^+ vec(K_\nu^e)$, where $D_\kappa^+ = (D_\kappa' D_\kappa)^{-1} D_\kappa'$, and D_κ is the

[1] If A is a full column rank matrix and B is conformable for the multiplication AB, the $rk(AB) = rk(B)$.

$\left(n^4 \times \frac{n(n+1)(n+2)(n+3)}{24}\right)$ matrix such that $D_\kappa \kappa_\nu^e = vec(K_\nu^e)$. For example, when $n = 2$, then:

$$D_\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the above vectorization, we have $\frac{\partial \kappa_\nu^e}{\partial \theta_i'} = D_\kappa^+ \frac{\partial vec(K_\nu^e)}{\partial vec(\Theta)'} \frac{\partial vec(\Theta)}{\partial \theta_i'}$ with $\frac{\partial vec(\Theta)}{\partial \theta_i'} = \Upsilon_{\theta_i}$. Given that equation (13) implies $vec(K_\nu^e) = [(\Theta \otimes \Theta \otimes \Theta) \otimes \Theta] vec(K_\epsilon^e)$, then $\frac{\partial vec(K_\nu^e)}{\partial vec(\Theta)'} = (I_{n^2} \otimes \Theta K_\epsilon^e) \frac{\partial vec(\Theta' \otimes \Theta' \otimes \Theta')}{\partial vec(\Theta)'} + [(\Theta' \otimes \Theta' \otimes \Theta') K_\epsilon^{e\prime} \otimes I_n]$, where $\frac{\partial vec(\Theta' \otimes \Theta' \otimes \Theta')}{\partial vec(\Theta)'} = (I_{n^2} \otimes C_{n,n^2} \otimes I_n) \left[(I_{n^4} \otimes vec(\Theta')) \frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} + [vec(\Theta' \otimes \Theta') \otimes I_{n^2}]\right] \frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$, and, as shown above, $\frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} = (I_n \otimes C_{n,n} \otimes I_n)[(I_{n^2} \otimes vec(\Theta')) + (vec(\Theta') \otimes I_{n^2})] \frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$ and $\frac{\partial vec(\Theta')}{\partial vec(\Theta)'} = C_{n,n}$. Note that $\frac{\partial \kappa_\nu^e}{\partial \theta_i'} = 0$ for $i = n\kappa$ in (18) and for $i = ss, ns\kappa$ in (19), since K_ν^e is not a function of the structural parameters relating the reduced-form innovations to the mesokurtic structural shocks. Moreover, $\frac{\partial \kappa_\nu^e}{\partial s_\epsilon} = D_\kappa^+ \frac{\partial vec(K_\nu^e)}{\partial vec(S_\epsilon)'} \frac{\partial vec(S_\epsilon)}{\partial s_\epsilon{}'}$, where $\frac{\partial vec(K_\nu^e)}{\partial vec(S_\epsilon)'} = 0$ given that K_ν^e is not a function of the skewnesses of the structural shocks. In addition, $\frac{\partial \kappa_\nu^e}{\partial \kappa_\epsilon^e} = D_\kappa^+ \frac{\partial vec(K_\nu^e)}{\partial vec(K_\epsilon^e)'} \frac{\partial vec(K_\epsilon^e)}{\partial \kappa_\epsilon^{e\prime}}$, where $\frac{\partial vec(K_\nu^e)}{\partial vec(K_\epsilon^e)'} = (\Theta \otimes \Theta \otimes \Theta \otimes \Theta)$ and $\frac{\partial vec(K_\epsilon^e)}{\partial \kappa_\epsilon^{e\prime}} = \Upsilon_{\kappa_\epsilon^e}$ is a $(n^4 \times m_\kappa)$ matrix selecting the partial derivatives with respect to the non-zero elements of κ_ϵ^e. For example, when $n = m_\kappa = 2$, then $\Upsilon_{\kappa_\epsilon^e}$ has values one for the (1,1) and (16,2) elements, and zero elsewhere. For the system with $n = 2$ and $m_\kappa = 1$, then $\Upsilon_{\kappa_\epsilon^e}$ has values one for the (1,1) element, and zero elsewhere.

Similarly to the case with skewed structural shocks, we can show that $rk[J_{\kappa_\nu^e,\theta}] = n \times m_\kappa$ and $rk[J_{\kappa_\nu^e,\kappa_\epsilon^e}] = m_\kappa$ for a full rank matrix Θ. In particular, the matrix $J_{\kappa_\nu^e,\theta_{\bullet,k}}$ has a form similar to the matrix $J_{s_\nu,\theta_{\bullet,k}}$ with elements function of θ_{lk}^3 on the diagonal of the block B_{lk}. Moreover, $rk\left[J_{\kappa_\nu^e,\theta} \; J_{\kappa_\nu^e,\kappa_\epsilon^e}\right] = n \times m_\kappa$ by noting that $J_{\kappa_\nu^e,\theta_{\bullet,k}} \times \theta_{\bullet,k} = 4\kappa_{\epsilon,kk,kk}^e J_{\kappa_\nu^e,\kappa_\epsilon^e,i}$ where k indexes the column of the respective matrix.

Appendix C: Rank condition

Let us now show that $rk[J] = r = r_s + r_{ns} + r_{s_\epsilon}$, as mentioned in appendix A. We need the following results for the rank of upper triangular block matrix :

Lemma 1 *Given that A is a $m \times n$ matrix, B is a $s \times t$ matrix and C is a $m \times t$ matrix,*

1.
$$rk(A) + rk(B) \leq rk\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) \leq rk(A) + rk\left(\begin{bmatrix} C \\ B \end{bmatrix}\right),$$

2.
$$rk(A) + rk(B) \leq rk\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) \leq rk\left(\begin{bmatrix} A & C \end{bmatrix}\right) + rk(B).$$

In Appendix B, it is shown that $rk\left[J_{s_\nu,\theta}\right] = rk\left[J_{s_\nu,\theta_s}\right] = n \times m_s$, $rk\left[J_{s_\nu,s_\epsilon}\right] = m_s$ and $rk\left[J_{s_\nu,\theta} \;\; J_{s_\nu,s_\epsilon}\right] = n \times m_s$. Moreover, each $\frac{n(n+1)}{2} \times n$ submatrix of $J_{\sigma_\nu,\theta}$ corresponding to each column of the matrix Θ is of rank equals to n. Now, we need to know the rank of the matrix of the derivative of the covariance matrix with respect to the parameters of the impact matrix J_{σ_ν,θ_s} and $J_{\sigma_\nu,\theta_{ns}}$. The rank of the first submatrix $rk[J'_{\sigma_\nu,\theta_s}] = \frac{n(n+1)}{2} - \frac{(n-m_s)(n-m_s+1)}{2}$ and for the second submatrix, the rank is equal to $rk[J'_{\sigma_\nu,\theta_{ns}}] = \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2}$. To understand this result, consider that $m_s = 1$. In this case, the $n \times n$ symmetric covariance matrix of the n-variables resulting from the skewed structural shock is of rank equals to one. Since only one row (column) is linear independent of the others rows (columns), this symmetric covariance matrix contains only n independent elements. The $n \times n$ symmetric covariance matrix of the n-variables resulting from the other structural shocks contains $n - m_s = n - 1$ linear independent rows (columns) which implies that this matrix has $n(n+1)/2 - 1$ idependent elements. For instance, suppose that $n = 3$ and $m_s = 1$ (where $\epsilon_{1,t}$ is the skewed structural shock), we get the following relationship:

$$\Sigma_\nu^{m_s} = \begin{bmatrix} \sigma^1_{\nu,11} & \sigma^1_{\nu,12} & \sigma^1_{\nu,13} \\ \sigma^1_{\nu,12} & \sigma^1_{\nu,22} & \sigma^1_{\nu,23} \\ \sigma^1_{\nu,13} & \sigma^1_{\nu,23} & \sigma^1_{\nu,33}. \end{bmatrix} = \begin{bmatrix} \theta_{11}^2 & \theta_{11}\theta_{21} & \theta_{11}\theta_{31} \\ \theta_{21}\theta_{11} & \theta_{21}^2 & \theta_{21}\theta_{31} \\ \theta_{31}\theta_{11} & \theta_{31}\theta_{21} & \theta_{31}^2 \end{bmatrix} = \begin{bmatrix} \theta_{11} \\ \theta_{21} \\ \theta_{31} \end{bmatrix} \begin{bmatrix} \theta_{11} & \theta_{21} & \theta_{31} \end{bmatrix} E(\epsilon_{1t}^2).$$

The rank of this matrix is equal to one because there is only one source of randomness; the skewed structural shock $\epsilon_{1,t}$. Consequently, only one row is linear independent of the other ones. This row contains n linear independent elements namely $\frac{n(n+1)}{2} - \frac{(n-m_s)(n-m_s+1)}{2} = 6 - 3 = 3$. The elements of the two other rows are linear combinations of this row. The rank of the symmetric covariance matrix for the n-variables induced by the two other structural shocks, denoted $\Sigma_\nu^{n-m_s}$, is :

$$\Sigma_\nu^{n-m_s} = \begin{bmatrix} \sigma^2_{\nu,11} & \sigma^2_{\nu,12} & \sigma^2_{\nu,13} \\ \sigma^2_{\nu,12} & \sigma^2_{\nu,22} & \sigma^2_{\nu,23} \\ \sigma^2_{\nu,13} & \sigma^2_{\nu,23} & \sigma^2_{\nu,33} \end{bmatrix}.$$

Since the rank of this submatrix is equal to the number of non-skewed structural shocks, there are two linear independent rows which contain $\frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} = 6-1 = 5$ independent elements for any combination of two rows of the matrix $\Sigma_\nu^{n-m_s}$. In the case where $m_s = 2$, there are two linear independent rows for the matrix $\Sigma_\nu^{m_s}$ which implies $\frac{n(n+1)}{2} - \frac{(n-m_s)(n-m_s+1)}{2} = 6-1 = 5$ independent elements and the matrix $\Sigma_\nu^{n-m_s}$ contains $\frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} = 6-1 = 3$ independent elements. As a result, the rank of Jacobian matrix $J_{\theta_s} = \begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} \end{bmatrix}'$ equals $n \times m_s$ by using $rk[J_{s_\nu,\theta_s}] = n \times m_s$ and $rk[J_{s_\nu,\theta_s}] \geq rk[J_{\sigma_\nu,\theta_s}]$. Now the rank of the Jacobian matrix $J_{\theta_{ns}} = \begin{bmatrix} J'_{\sigma_\nu,\theta_{ns}} & J'_{s_\nu,\theta_{ns}} \end{bmatrix}'$ is equal to the rank of the Jacobian matrix $J_{\sigma_\nu,\theta_{ns}}$ which is $\frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2}$ since $J_{s_\nu,\theta_{ns}}$ is a matrix of zeros. Finally, the rank of the matrix $J_{s_\epsilon} = \begin{bmatrix} J'_{\sigma_\nu,s_\epsilon} & J'_{s_\nu,s_\epsilon} \end{bmatrix}'$ is equal to the rank of the matrix J_{s_ν,s_ϵ} because only the coskewness matrix gives information about the third moment of the structural shocks. The rank of J_{s_ϵ} is $rk\left(J_{s_\nu,s_\epsilon}\right) = m_s$. The rank of the complete matrix of the Jacobian J respective to the structural parameters :

$$J = \begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & 0 \\ J_{s_\nu,\theta_s} & 0 & J_{s_\nu,s_\epsilon} \end{bmatrix} \tag{C.1}$$

can then be shown to be equal to $rk[J] = r = r_s + r_{ns} + r_{s_\epsilon}$, where $r_s = n \times m_s$, $r_{ns} = \frac{n(n+1)}{2} - \frac{m_s(m_s+1)}{2}$ and $r_{s_\epsilon} = m_s$. First, consider the rank of the following block diagonal submatrix

$$\begin{bmatrix} J_{\sigma_\nu,\theta_{ns}} & 0 \\ 0 & J_{s_\nu,s_\epsilon} \end{bmatrix}. \tag{C.2}$$

The rank of this submatrix equals the sum of the rank of the block diagonal submatrices, namely $rk(J_{\sigma_\nu,\theta_{ns}}) + rk(J_{s_\nu,s_\epsilon}) = \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} + m_s$.

Second, the rank of (C.1) equal the rank of (C.2) plus the rank of J_{θ_s} except if there exists at least one linear combination of the columns from the matrix J_{θ_s} which corresponds to a column of (C.2). In the following, it is shown that such linear combination does not exist for a full rank matrix Θ. We show that such linear combination does not exist in two steps : i) there is no linear combination of J_{θ_s} which yields a column of $J_{\theta_{ns}}$ and ii) there is no linear combination of J_{θ_s} which yields a column of J_{s_ϵ}. For i), consider the submatrix $[J_{\theta_s} \;\; J_{\theta_{ns}}]$ which is

$$J_\theta = \begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} \\ J_{s_\nu,\theta_s} & 0 \end{bmatrix}.$$

The rank of J_θ equal to the rank of the submatrix J_{s_ν,θ_s} plus the rank of the submatrix $J_{\sigma_\nu,\theta_{ns}}$. Thus $rk(J_\theta) = n \times m_s + \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2}$. Indeed, the rank of the bloc matrix J_θ is equal to the rank of the matrix $\begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} \end{bmatrix}'$ plus the rank of the matrix $J_{\sigma_\nu,\theta_{ns}}$ using the following inequalities for the rank of upper triangular block matrix (Lemma 1):

$$rk(J_{\sigma_\nu,\theta_{ns}}) + rk(J_{s_\nu,\theta_s}) \leq rk(J_\theta) \leq rk(J_{\sigma_\nu,\theta_{ns}}) + rk\left(\begin{bmatrix} J_{\sigma_\nu,\theta_s} \\ J_{s_\nu,\theta_s} \end{bmatrix}\right).$$

Here, we have

$$rk(J_{\sigma_\nu,\theta_{ns}}) + rk(J_{s_\nu,\theta_s}) = rk(J_{\sigma_\nu,\theta_{ns}}) + rk\left(\begin{bmatrix} J_{\sigma_\nu,\theta_s} \\ J_{s_\nu,\theta_s} \end{bmatrix}\right).$$

For ii), we show that there is no linear combination of J_{θ_s} that yields a column of J_{s_ϵ}. In the preceding section, we show that $rk\left[J_{s_\nu,\theta_s} \;\; J_{s_\nu,\theta_{ns}} \;\; J_{s_\nu,s_\epsilon}\right] = rk\left[J_{s_\nu,\theta_s}\right]$ which implies that it exists an appropriated matrix A of dimension $(n \cdot m_s) \times m_s$ such that $\left[J_{s_\nu,\theta_s}\right] A = J_{s_\nu,s_\epsilon}$ since the submatrix $J_{s_\nu,\theta_{ns}} = \mathbf{0}$ is a matrix of zeros. Define each column of the matrix A by A_i for $i = 1, \ldots, m_s$.[2] For a matrix Θ of full rank, all $\frac{n(n+1)}{2} \times n$ submatrices $\left[J_{\sigma_\nu,\theta_{i,s}}\right]$ are necessarily of full rank so there is no vector such as $\left[J_{\sigma_\nu,\theta_{i,s}}\right] A_i = 0$ for $\forall i$ where i indexes the elements of the vector θ_s corrresponding to the column i of the matrix Θ_s. This implies that the rank of the matrix J equals $n \times m_s + \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} + m_s$. Given that $\left[J_{\sigma_\nu,\theta_{i,s}}\right] A_i \neq 0$ for $i = 1, \ldots, m_s$ and that J_{s_ν,θ_s} is of full rank, there is no linear combination of the columns of the matrix J_{θ_s} that that corresponds to a column of the matrix (C.2) since the Jacobian matrix respective of the structural parameter J_θ is of full rank. This completes the proof.

The same results hold for the case which exploits only the fourth moments of the structural shocks by modifying properly the dimension of the matrices and the notation.

For the general case

$$J = \begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} & J_{\sigma_\nu,\theta_{ns\kappa}} & J_{\sigma_\nu,s_\epsilon} & J_{\sigma_\nu,\kappa^e_\epsilon} \\ J_{s_\nu,\theta_{ss}} & J_{s_\nu,\theta_{\kappa\kappa}} & J_{s_\nu,\theta_{s\kappa}} & J_{s_\nu,\theta_{ns\kappa}} & J_{s_\nu,s_\epsilon} & J_{s_\nu,\kappa^e_\epsilon} \\ J_{\kappa^e_\nu,\theta_{ss}} & J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} & J_{\kappa^e_\nu,\theta_{ns\kappa}} & J_{\kappa^e_\nu,s_\epsilon} & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}$$

which equals

$$J = \begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} & J_{\sigma_\nu,\theta_{ns\kappa}} & 0 & 0 \\ J_{s_\nu,\theta_{ss}} & 0 & J_{s_\nu,\theta_{s\kappa}} & 0 & J_{s_\nu,s_\epsilon} & 0 \\ 0 & J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} & 0 & 0 & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix} \tag{C.3}$$

First, consider the block diagonal submatrix containing the last subgroup of columns

$$\begin{bmatrix} J_{\sigma_\nu,\theta_{ns\kappa}} & 0 & 0 \\ 0 & J_{s_\nu,s_\epsilon} & 0 \\ 0 & 0 & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}. \tag{C.4}$$

The rank of this submatrix equals the sum of the rank of the block diagonal submatrices, $rk(J_{\sigma_\nu,\theta_{ns\kappa}}) + rk(J_{s_\nu,s_\epsilon}) + rk(J_{\kappa^e_\nu,\kappa^e_\epsilon}) = \frac{n(n+1)}{2} - \frac{(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})(m_{ss}+m_{\kappa\kappa}+m_{s\kappa}+1)}{2} + m_s + m_\kappa$.

By an argument similar to the one above, the rank of the submatrix

$$\begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} \\ J_{s_\nu,\theta_{ss}} & 0 & J_{s_\nu,\theta_{s\kappa}} \\ 0 & J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} \end{bmatrix} \tag{C.5}$$

[2]From Appendix B, A_i corresponds to the column of matrix θ_s divided by 3 times the respective measure of skewness.

equals the sum of rank of the submatrix $\begin{bmatrix} J'_{\sigma_\nu,\theta_{ss}} & J'_{s_\nu,\theta_{ss}} \end{bmatrix}'$ and the rank of $\begin{bmatrix} J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} \end{bmatrix}$, using Lemma 1 and the fact that $rk\begin{bmatrix} J_{\theta_{\kappa\kappa}} & J_{\theta_{s\kappa}} \end{bmatrix} = rk\begin{bmatrix} J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} \end{bmatrix} = n \times m_{\kappa\kappa} + n \times m_{s\kappa}$. The rank of (C.5) is then $n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa}$. Now, one needs to show that the rank of the complete Jacobian matrix (C.3) is the sum of the rank of (C.4) and (C.5). First, the rank of the submatrix containing (C.5) and $\begin{bmatrix} J'_{\sigma_\nu,\theta_{ns\kappa}} & 0' & 0' \end{bmatrix}'$ equals the rank of (C.5) plus the rank of J_{σ_ν} by the lower triangular block structure of this submatrix (by Lemma 1) which is $n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa} + \frac{n(n+1)}{2} - \frac{(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})(m_{ss}+m_{\kappa\kappa}+m_{s\kappa}+1)}{2}$. By a proof similar to the one to the case under asymmetry only, for a full rank matrix Θ, there is no linear combination of (C.5) that can yield a column of the last two submatrices of (C.4), i.e.

$$\begin{bmatrix} 0 & 0 \\ J_{s_\nu,s_\epsilon} & 0 \\ 0 & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}.$$

The rank of J is then equals to $rk[J_{\theta_{ss}}] + rk[J_{\theta_{\kappa\kappa}}] + rk[J_{\theta_{s\kappa}}] + rk[J_{\theta_{ns\kappa}}] + rk[J_{s_\nu}] + rk[J_{\kappa^e_\epsilon}] = n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa} + \left(\frac{n(n+1)}{2} - \frac{(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})(m_{ss}+m_{\kappa\kappa}+m_{s\kappa}+1)}{2}\right) + m_s + m_k$.

Finally, Corollary 1 results from that there is no linear combination of (C.5) that can yield a column of the last two submatrices of (C.4)

Appendix D: Asymptotic Distribution of the Rank Test

First, we derive the asymptotic distribution of the statistics $\widehat{CRT}^{LR}_{r^*}$ and $\widehat{CRT}^{W}_{r^*}$. Under the assumption in section 3.1 for K^e_ϵ, $E[\|\epsilon_t\|^8] < \infty$ and the estimator $\widehat{K}^e_u$ is a root-T consistent for the $n \times n^3$ excess cokurtosis matrix K^e_u of the normalized reduced-form innovations. In this context, the asymptotic distribution of $\widehat{K}^e_u$ is

$$T^{1/2} vec(\widehat{K}^e_u - K^e_u) \xrightarrow{\mathcal{L}} N(0, \Gamma)$$

where Γ is finite.

Now, suppose that the matrix K^e_u is of rank $r^* \leq n$. The singular value decomposition of K^e_u gives $K^e_u = C\Lambda D'$ where Λ is a diagonal matrix with the singular values on the diagonal. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the singular values of the matrix Λ ordered in decreasing values. For a matrix K^e_u of rank equal to r^*, the first r^* singular values are different from zero and the last $n - r^*$ singular values are equal to zero. Thus

$$C' K^e_u D = \begin{bmatrix} C'_{r^*} K^e_u D_{r^*} & C'_{r^*} K^e_u D_{n^3-r^*} \\ C'_{n-r^*} K^e_u D_{r^*} & C'_{n-r^*} K^e_u D_{n^3-r^*} \end{bmatrix} = \Lambda.$$

The submatrix $C'_{n-r^*} K^e_u D_{n^3-r^*}$ corresponds to the null space of K^e_u which is the object of interest (see Al-Sadoon, 2017). We have

$$\sum_{i=r^*+1}^{n} \hat{\lambda}_i^2 = \|vec(\widehat{C}'_{n-r^*} \widehat{K}^e_{u,c} \widehat{D}_{n^3-r*})\|^2 = \|vec(\widehat{C}_{n-r^*} \widehat{C}'_{n-r^*} \widehat{K}^{e,b}_{u,c} \widehat{D}_{n^3-r*} \widehat{D}'_{n^3-r*})\|^2$$

where $U_{n-r^*} = C_{n-r^*}C'_{n-r^*}$ and $V_{n^3-r^*} = D_{n^3-r^*}D'_{n^3-r^*}$ are the orthogonal projectors onto the space spanned by the left and the right null space singular vectors.[3]

The vectorization of this matrix yields

$$vec\left(\widehat{U}_{n-r^*}\widehat{K}_u^e\widehat{V}_{n^3-r^*}\right) = \left(\widehat{V}_{n^3-r^*} \otimes \widehat{U}_{n-r^*}\right) vec(\widehat{K}_u^e).$$

Since $T^{1/2}vec(\widehat{K}_u^e - K_u^e) \to N(0, \Gamma)$, the convergence in probability of the orthogonal projectors $\widehat{U}_{n-r^*} \xrightarrow{\mathcal{P}} U_{n-r^*}$ and $\widehat{V}_{n^3-r^*} \xrightarrow{\mathcal{P}} V_{n^3-r^*}$[4] and $\widehat{\Gamma} \xrightarrow{\mathcal{P}} \Gamma$, this implies that

$$T^{1/2}\left(\widehat{V}_{n^3-r^*} \otimes \widehat{U}_{n-r^*}\right)' vec(\widehat{K}_u^e - K_u^e) \xrightarrow{\mathcal{L}} N\left(0, \left(V_{n^3-r^*} \otimes U_{n-r^*}\right)\Gamma\left(V_{n^3-r^*}. \otimes U_{n-r^*}\right)\right)$$

Statistics $\widehat{CRT}_{r^*}^{LR}$ and $\widehat{CRT}_{r^*}^{W}$ converge asymptotically to

$$Tr(X_{r^*}X'_{r^*}) + o_p(1) = vec(X_{r^*})'vec(X_{r^*}) + o_p(1)$$

where $X_{r^*} = T^{1/2}\left(V'_{n^3-r^*} \otimes U'_{n-r^*}\right) vec(\widehat{K}_u^e - K_u^e)$. Both statistics have a limiting distribution given by $\sum_{i=1}^{t^*} \delta_i^{r^*} Z_i^2$ where $\delta_1^{r^*} \geq \ldots \geq \delta_{t^*}^{r^*}$ are the non-zero ordered eigenvalues of the matrix $\left(V_{n^3-r^*} \otimes U_{n-r^*}\right)\Gamma\left(V_{n^3-r^*} \otimes U_{n-r^*}\right)$ and $\{Z_i\}_{i=1}^{t^*}$ are independent $N(0,1)$ variates. The limiting distribution is then a weighted sum of t^* independent chi-squared variables with one degree of freedom and the weights are given by the non-zero eigenvalues $\delta_i^{r^*}$ for $i = 1, \ldots, t^*$. An estimator of the cumulative distribution function is obtained using the estimated counterparts of the matrices U_{n-r^*}, $V_{n^3-r^*}$ and Γ and the c.d.f. of the corresponding weighted sum of Z_i^2 for $i = 1, \ldots, t^*$ which can be easily evaluated by simulation.

Now we show that the subvector $u_{r^*,t}^b$ obtained by bootstrapping the vector $\omega_{r^*,t}^{b'} = \widehat{C}'_{r^*}\hat{u}_t$ for $b = 1, \ldots, B$ implies that $\hat{\lambda}_i^b \xrightarrow{\mathcal{P}} \hat{\lambda}_i$ where $\hat{\lambda}_i^b$ are the bootstrap estimators of the r^* largest singular values and $\hat{\lambda}_i$ are the sample estimators. Suppose a vector z with the following relation with a vector u:

$$z_t = C'u_t$$

where C is orthonormal. We have the following relation for the excess cokurtosis

$$K_z^e = C'K_u^e\left(C \otimes C \otimes C\right)$$

[3]Unlike to Robin and Smith (2000) and Bura and Yang (2011) but similarly to Portier and Delyon (2014), we consider orthogonal projection matrices U_{n-r^*} and $V_{n^3-r^*}$. The orthogonal projection matrices are invariant to the choice of a basis while the singular vectors in C_{n-r^*} and $D_{n^3-r^*}$ are uniquely defined only up to post-multiplication by an orthogonal matrix in a case of a multiplicity of singular values. Moreover, the orthogonal projection is continuous in the elements of the matrix, a necessary condition to guarantee the convergence in probability (see Dufour and Valéry, 2012).

[4]See Al-Sadoon, 2017, Theorem 1.

For the quadratic form of the excess cokurtosis

$$K_z^e K_z^{e\prime} \;\; = \;\; C' K_u^e \left(C \otimes C \otimes C\right) \left(C' \otimes C' \otimes C'\right) K_u^{e\prime} C = C' K_{u,c}^e K_u^{e\prime} C.$$

By the eigenvalue decomposition $K_{u,}^e K_u^{e\prime} = C\Lambda^2 C'$ which implies $K_z^e K_z^{e\prime} = \Xi = diag(\lambda_1^2, \ldots, \lambda_{r*}^2, 0, \ldots, 0)$ for a matrix K_u^e of rank $r*$ with the eigenvalues in descending order, where the eigenvalues are the square of the singular values λ_i. Thus, linear combinations of the normalized reduced-form innovations $\omega_{r^*} = \widehat{C}'_{r^*} \hat{u}_t$ capture the excess cokurtosis of the vector of the normalized reduced-form innovations where $\widehat{C}_r^*$ are the first r^* columns of $\widehat{C}$ corresponding to the singular values $\lambda_1, \ldots, \lambda_{r^*}$. The subvector $u_{r^*,t}^b$ is generated by bootstrapping the vector $\omega'_{r^*,t} = \widehat{C}'_{r^*} \hat{u}_t$ for $b = 1, \ldots, B$. Thus, for a consistent estimator of the excess cokurtosis $\widehat{K}^e_{u^b_{r^*}}$ of $u_{r^*,t}^b$ for $b = 1, \ldots, B$, a given matrix $\widehat{C}_{r^*}$ and by the continuity of the singular values, $\hat{\lambda}_i^b(\widehat{K}^e_{u^b_{r^*}} \widehat{K}^{e\prime}_{u^b_{r^*}}) \xrightarrow{\mathcal{P}} \hat{\lambda}_i(\widehat{C}'_{r^*} \widehat{K}_u^e \widehat{K}_u^{e\prime} \widehat{C}_{r^*})$ for $i = 1, \ldots, r^*$.

Appendix E: Empirical sizes and powers of rank tests for symmetry

This appendix reports the empirical sizes and powers of rank tests for symmetry. Table E.1 shows the empirical sizes. The Wald test with asymptotic distributions has empirical sizes that slightly deviate from the nominal ones, and the likelihood-ratio test with limiting distributions has empirical sizes that are substantially smaller than the nominal counterparts. In contrast, both the Wald and likelihood-ratio tests with finite-sample distributions feature empirical sizes that are almost identical to the nominal sizes, regardless of the number of observations in the sample.

Table E.2 displays the empirical powers. For the Wald and likelihood-ratio tests with finite-sample distributions, the powers substantially improve as the sample size increases and as the structural shocks become more skewed.

Table E.1. Empirical Sizes of Rank Tests: Skewness

	Asymptotic Distributions						Finite-Sample Distributions					
	$r^*=0$						$r^*=0$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	8.72	3.92	0.53	2.68	0.63	0.01	9.42	4.65	0.98	9.56	4.85	1.01
200	9.99	4.66	0.80	5.81	1.91	0.12	10.17	5.25	0.98	10.19	5.20	1.00
500	9.93	4.69	0.81	7.97	3.36	0.41	10.14	5.04	1.10	10.29	4.99	1.12
1,000	9.73	4.63	0.70	8.65	3.94	0.52	9.82	4.91	0.92	9.87	4.90	0.92
5,000	10.03	5.22	1.09	9.90	4.97	1.02	10.02	5.10	1.12	9.98	5.11	1.11
	$r^*=1$						$r^*=1$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	11.83	5.79	1.52	7.86	3.22	0.51	11.41	6.35	1.47	11.41	6.35	1.47
200	10.87	5.30	1.18	8.60	3.66	0.53	9.11	4.86	1.42	9.11	4.86	1.42
500	10.89	5.20	1.06	9.74	4.42	0.63	9.29	4.55	1.07	9.29	4.55	1.07
1,000	9.97	4.82	1.03	9.45	4.36	0.86	8.39	4.26	1.02	8.39	4.26	1.02
5,000	10.61	5.59	1.02	10.05	5.47	0.99	9.20	4.68	0.96	9.20	4.68	0.96

Notes. Entries are the empirical sizes (in percentage) of the rank tests with asymptotic and finite-sample distributions under the null hypothesis that $rk[S_u]=r^*$. The empirical sizes are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_1=-0.5$, $\alpha_2=0.5$ and $\omega_1=\omega_2=1$. Also, the distributions are $\epsilon_{2,t}\sim N(0,1)$, and i) $\epsilon_{1,t}\sim N(0,1)$ under $r^*=0$ or ii) $2.1755\times\epsilon_{1,t}\sim N(1,1)$ with probability 0.7887 and $2.1755\times\epsilon_{1,t}\sim N(-3.7326,1)$ with probability 0.2113 under $r^*=1$. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}^{W}_{r^*}$ and the likelihood-ratio (LR) statistic $\widehat{CRT}^{LR}_{r^*}$ associated with S_u exceed the critical values. The asymptotic critical values are computed as shown in Appendix D. The finite-sample critical values are computed by the bootstrap procedure elaborated in Section 4.2.

Table E.2. Empirical Powers of Rank Tests with Finite-Sample Distributions: Skewness

	$Skewness = -0.5231$						$Skewness = -0.9907$					
			$r^* = 0$						$r^* = 0$			
		Wald			LR			Wald			LR	
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	20.71	11.44	2.42	20.88	11.46	2.53	72.05	46.66	10.43	69.95	44.82	10.53
200	41.02	26.70	8.50	40.58	26.40	8.15	99.35	96.85	74.28	99.23	96.33	67.90
500	82.98	71.28	42.66	82.82	70.93	41.24	100.0	100.0	100.0	100.0	100.0	100.0
$1,000$	99.11	97.66	88.94	99.10	97.64	88.51	100.0	100.0	100.0	100.0	100.0	100.0
			$r^* = 1$						$r^* = 1$			
		Wald			LR			Wald			LR	
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	16.35	8.05	1.31	16.35	8.05	1.31	88.27	78.73	41.91	89.15	78.75	41.91
200	41.12	27.24	8.06	41.12	27.24	8.06	99.70	99.20	94.65	99.70	99.20	94.65
500	86.85	78.10	53.80	86.85	78.10	53.80	100.0	100.0	100.0	100.0	100.0	100.0
$1,000$	99.49	98.65	94.17	99.49	98.65	94.17	100.0	100.0	100.0	100.0	100.0	100.0

Notes. Entries are the empirical powers (in percentage) of the rank tests with finite-sample distributions under the null hypothesis that $rk[S_u] = r^*$. The empirical powers are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_1 = -0,5$, $\alpha_2 = 0.5$ and $\omega_1 = \omega_2 = 1$. For $r^* = 0$, the distributions are: i) $\epsilon_{2,t} \sim N(0,1)$ as well as $1.6808 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.5 and $1.6808 \times \epsilon_{1,t} \sim N(-1, 2.65)$ with probability 0.5 when $\epsilon_{1,t}$ exhibits a skewness of -0.5231, and ii) $\epsilon_{2,t} \sim N(0,1)$ as well as $2.1755 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{1,t} \sim N(-3.7326, 1)$ with probability 0.2113 when $\epsilon_{1,t}$ exhibits a skewness of -0.9907. For $r^* = 1$, the distributions are: i) $1.6808 \times \epsilon_{2,t} \sim N(1,1)$ and $1.6808 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.5 as well as $1.6808 \times \epsilon_{2,t} \sim N(-1, 2.65)$ and $1.6808 \times \epsilon_{1,t} \sim N(-1, 2.65)$ with probability 0.5 when each shock exhibits a skewness of -0.5231, and ii) $2.1755 \times \epsilon_{2,t} \sim N(1,1)$ and $2.1755 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.7887 as well as $2.1755 \times \epsilon_{2,t} \sim N(-3.7326, 1)$ and $2.1755 \times \epsilon_{1,t} \sim N(-3.7326, 1)$ with probability 0.2113 when each shock exhibits a skewness of -0.9907. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}^{W}_{r*}$ and the likelihood-ratio (LR) statistic $\widehat{CRT}^{LR}_{r*}$ associated with S_u exceed the finite-sample critical values, where the latters are computed by the bootstrap procedure elaborated in Section 4.2.

Appendix F: Estimates of the structural parameters

Table F.1 shows the estimates of the structural parameters involved in system (31).

Table F.1. Parameter Estimates

Parameter	$\alpha_2 = 0$	$\alpha_1 = 2.08$	$\beta_1 = 0$
α_1	1.9409***	2.0800†	1.8359**
α_2	0.0000†	−0.5711*	0.0728
β_1	0.3797**	−0.1482*	0.0000†
β_2	−0.0015	0.0095*	−0.0030
γ_1	−0.0013	−0.0021	0.0002
γ_2	0.0439	0.3235***	0.2516***
ω_τ	0.0474***	0.0473***	0.0474***
ω_g	0.0064***	0.0071***	0.0068***
ω_y	0.0050***	0.0048***	0.0048***
$\kappa^e_{\epsilon,11,11}$	2.8284***	2.8135***	2.8114***

Notes. Entries correspond to the estimates of the parameters of system (31). $*$, $**$, and $***$ indicate, respectively, that the 90, 95, and 99 percent confidence interval does not include zero, where the confidence intervals are computed from 5,000 bootstrap samples. $\dagger$ indicates that the parameter is constrained. The restrictions $\alpha_2 = 0$, $\alpha_1 = 2.08$, and $\beta_1 = 0$ imply that $\theta_{12} = \alpha_1\theta_{32}$, $\theta_{13} = \alpha_1\theta_{33}$, and $\theta_{23} = 0$.

References

[1] Al-Sadoon (2017), “A Unifying Theory of Tests of Rank, ” *Journal of Econometrics* 199, pp. 49–62.

[2] Bura, E., and J. Yang (2011), “Dimension Estimation in Sufficient Dimension Reduction: a Unifying Approach ” *Journal of Multivariate Analysis* 102, pp. 130–142.

[3] Dufour, J.M.and P. Valéry (2012), “Wald-Type Test When Rank Conditions Fail: a Smooth Regularization Approach, ” CIRANO Working Paper.

[4] Lütkepohl, H. (2007), *New Introduction to Multiple Time Series Analysis,* Berlin: Springer, 764 p.

[5] Magnus, J.R., and H. Neudecker (2007), *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Third Edition, New York: John Wiley & Sons, 450 p.

[6] Portier, F., and B. Delyon (2014), “Bootstrap Testing of the Rank of a Matrix via Least Squared Constrained Estimation,” *Journal of the American Statistical Association* 109, pp. 160–172.

[7] Robin, J.-M., and R.J. Smith (2000), “Tests of Rank,” *Econometric Theory* 16*,* pp. 151–175.