Identification of Structural Vector Autoregressions Through Higher Unconditional Moments*

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Abstract

This paper pursues two objectives. First, we determine the sufficient condition for local, statistical identification of SVAR processes through the third and fourth unconditional moments of the reduced-form innovations. Our findings provide novel insights when the entire system is not identified, as they highlight which subset of structural parameters is identified and which is not. Second, we elaborate a tractable testing procedure to verify whether the identification condition holds, prior to the estimation of the structural parameters of the SVAR process. To do so, we design a new bootstrap procedure that improves the small-sample properties of rank tests for the symmetry and kurtosis of the structural shocks.

JEL classification: C12, C32, C51.

Keywords: Bootstrap procedure, excess kurtosis, identification condition, rank test, skewness, structural vector autoregression.

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1. Introduction

Econometric methods for simultaneous equation models highlight the importance of verifying the identification before proceeding to the estimation of the structural parameters. Namely, it is only after verifying global and local identification that it becomes feasible to estimate all the structural parameters. In this vein, this paper pursues two objectives. First, we derive a sufficient condition for local, statistical identification of Structural Vector Autoregressive (SVAR) processes through higher unconditional moments. Second, we develop a tractable method to verify whether a SVAR process is identified, prior to the estimation of the structural parameters.

A first strand of the SVAR literature relies on the standard assumption that the structural shocks are orthogonal and extracts the information contained in the unconditional covariances of the reduced-form innovations to identify the structural parameters. As is well known, this information is insufficient to identify all the parameters, so that short-run restrictions (e.g. Sims, 1980), long-run restrictions (e.g. Blanchard and Quah, 1989), and/or sign restrictions (e.g. Uhlig, 2005) need to be placed. If the restrictions are economically motivated, then the imposition of enough restrictions gives rise to economic identification in the sense that the dynamic responses become interpretable given that the structural shocks are economically meaningful. However, it is not possible to verify jointly the validity of all the restrictions by applying formal statistical tests.

A second strand of the literature exploits the information related to certain statistical properties of the data, in addition to the unconditional covariances of the reduced-form innovations (see Kilian and Lütkepohl, 2017, Chapter 14). If this information is rich enough then this strategy yields local identification, without resorting to any identifying restrictions, and, hence, the dynamic response matrices are unique up to changes in sign and permutations of columns. It also produces statistical (rather than economic) identification as nothing guarantees that the dynamic responses and structural shocks have an economic interpretation. In this framework, it is possible to verify the validity of certain classes of restrictions (e.g. short- and long-run restrictions) that would have been required if only the unconditional covariances of the reduced-form innovations were taken into account. This is convenient, for example, to formally select among alternative sets of restrictions reflecting competing economic theories.

One method relying on the statistical properties of the data specifies the time-varying variances of the structural shocks, while preserving the standard assumption that theses shocks are orthogonal. In this context, all the structural parameters involved in the SVAR are identified, without placing any restrictions, when all but one, structural shocks display distinct time-varying variances. Note, however, that the method requires to take a stand about whether the time-varying variances

are determined by fixing a priori the dates of the structural breaks, are specified via GARCH processes, or are modeled by regime switching processes with Markov chains or smooth transitions (e.g. Rigobon, 2003; Normandin and Phaneuf, 2004; Lanne et al., 2010; Lütkepohl and Netšunajev, 2014; Lütkepohl and Schlaak, 2018).[1] Another approach is based on unconditional non-normal distributions of the structural shocks, but assumes that these shocks are independent. In this environment, all the structural parameters are identified, when all but one, structural shocks are non-normally distributed (see Comon, 1994; Eriksson and Koivunen, 2004; Herwartz, 2015; Gouriéroux et al., 2017; Funovits, 2019).[2] Observe, however, that the assumption of independent structural shocks is more restrictive than the standard one stating that the shocks are orthogonal; that is, it is not always possible to recover independent structural shocks from non-normal reduced-form innovations through linear transformations (see Kilian and Lütkepohl, 2017, Chapter 14).

A key goal of this paper is to determine the local, statistical identification conditions of SVAR processes through the third and fourth unconditional moments of the reduced-form innovations.[3] For this purpose, we assume that the structural shocks display zero cross-sectional covariances, coskewnesses, and excess cokurtoses.[4] Note that this can be viewed as a natural extension to the third and fourth unconditional comoments of the standard assumption that the structural shocks are orthogonal. Moreover, our assumption admits the possibility that the structural shocks exhibit time-varying conditional variances (although we do not need to specify the process governing these variances) and is milder than the assumption stating that the shocks are independent.[5] In our context, not only the covariances of the reduced-form innovations, but also the coskewnesses and excess cokurtoses of these innovations can be exploited to identify extra structural parameters, and, hence, to relax some of the identifying restrictions required when the information contained in the third and fourth moments is ignored. Formally, we derive a sufficient (rank) condition for local, statistical identification by extending the developments of Lütkepohl (2007, Chapter 9). We further express this rank condition in terms of simple formulas, which exclusively involve the numbers of

[1] Lewis (2019) proposes an identification strategy based on time-varying volatility of general form without specifying a particular parametric model.

[2] Alternatively, Gouriéroux et al. (2018) show that all structural parameters are identified under the assumptions that the reduced-form innovations are strong white noises and the structural shocks are mutually independent and have finite moments of order four.

[3] Gospodinov and Ng (2015) also consider third- and fourth-order cumulants for the identification and estimation of possibly nonivertible moving average models.

[4] We assume that the first four unconditional moments of the structural shocks exist. This assumption is commonly invoked to demonstrate that the ordinary least square and maximum likelihood estimators of VAR parameters are consistent and asymptotically normal, and that bootstrap inference is valid (see Lütkepohl, 2007, Chapter 3; Kilian and Demiroglu, 2000). This assumption is also used in the specification of the pseudo likelihood function associated with SVAR processes with independent structural shocks (e.g. Gouriéroux et al., 2017).

[5] See also Kilian and Lütkepohl, 2017, Chapter 14, page 514, for an example where the structural shocks display zero cross-sectional covariances and excess cokurtoses but they are not mutually independent.

structural shocks displaying non-zero skewnesses and excess kurtoses. Given this information, it is most easy for empirical researchers to determine whether or not the structural system is identified.

Our results regarding the identification of the entire structural system parallel the existing results. That is, all the structural parameters are identified when all but one, structural shocks exhibit non-zero skewnesses and/or excess kurtoses (see Comon, 1994 and Keweloh, 2020). Our findings further provide novel insights when the entire SVAR process is not identified by the second, third and fourth unconditional moments as they highlight which subset of structural parameters is identified and which is not according to the sufficient condition. This leads to three important implications. First, one can establish which structural subsystem is identified. Note that this subsystem generates the dynamic responses of all the variables included in the SVAR process to the structural shocks which are asymmetric and/or non-mesokurtic.[6] Second, one can determine the structural parameters on which some restrictions must be placed on in order to achieve the identification of the entire system. Third, one can test the validity of economic and statistical restrictions (by treating these as overidentifying restrictions) that are commonly placed on the structural subsytem that is identified through higher unconditional moments.

Another prime aim of this paper is to elaborate a tractable testing procedure to verify whether the identification condition holds, prior to the estimation of the structural parameters involved in the SVAR process. As stated above, verifying our identification condition requires the knowledge of the numbers of asymmetric and non-mesokurtic structural shocks. At first glance, this may seem problematic for practitioners, as the structural shocks become measurable only once the structural system is estimated.[7] However, we demonstrate that the numbers of structural shocks displaying non-zero skewnesses and excess kurtoses correspond to the ranks of the coskewness and excess cokurtosis matrices of the reduced-form innovations, where these matrices are easily constructed from sample estimates of the moments of the reduced-form residuals — without having to proceed to the estimation of the structural system.

In this paper, we design a new bootstrap procedure to approximate the finite-sample distributions in order to test the ranks of the coskewness and excess cokurtosis matrices of the reduced-form innovations. We show that this procedure allows to overcome size distortions. Specifically, both the Wald and likelihood-ratio tests with bootstrap critical values feature empirical sizes that are almost

[6] For briefness, throughout the text symmetric (asymmetric) and mesokurtic (non-mesokurtic) variables refer to variables with symmetric (asymmetric) and mesokurtic (non-mesokurtic) distributions. Also, a symmetric (asymmetric) distribution implies a zero (non-zero) skewness, whereas a mesokurtic (non-mesokurtic) distribution implies a zero (non-zero) excess kurtosis.

[7] As a result, existing studies do not verify whether the structural shocks are asymmetric or non-mesokurtic before proceeding to the estimation of the structural system; see for example Moneta et al. (2013), Lanne et al. (2017), Gouriéroux et al. (2017), Lanne and Luoto (2019) and Keweloh (2020). They instead verify whether the reduced-form innovations are asymmetric or non-mesokurtic.

identical to the nominal ones, regardless of the number of observations in the sample. In sharp contrast, the tests with asymptotic distributions have empirical sizes that are often substantially smaller than the nominal counterparts, even for large samples.

Finally, we illustrate our developments by identifying the effects of fiscal policies on economic activity; a topic that has received renewed interest in light of the recent Great Recession. For this purpose, we perform the analysis on a trivariate SVAR process which includes taxes, public spending, and output for the U.S. The empirical results for the Wald and likelihood-ratio bootstrap versions for the rank tests indicate that all the structural shocks are symmetric and only one structural shock is non-mesokurtic. Based on this information, the identification condition and the estimation results reveal that the subsytem relating all the variables to the tax shock is identified. In contrast, the subsytem relating the variables to the public spending shock is under-identified. Also, we show that the restrictions invoked in the seminal study of Blanchard and Perotti (2002) imply that the subsytem relating the variables to the spending shock becomes over-identified. We further document that the effects of the spending shock highly depend on the nature of the identifying restrictions used.

Recently, third and fourth unconditional moments are also be used to identify and estimate the entire system of a SVAR. Lanne and Luoto (2019) have proposed a GMM estimator of SVAR which primarly relies on the excess kurtosis of the structural shocks. As our proposed identification condition, they do not assume the structural shocks to be mutually independent. The GMM estimator proposed by Lanne and Luoto (2019) uses a sufficient number of moment restrictions based on subset of cokurtosis conditions implied by independence. The assumption of zero cross-sectional coskewnesses and excess cokurtoses of the structural shocks is not necessary to achieve the local identification of the entire SVAR process. Indeed, a sufficient subset of these restrictions related to the excess cokurtosis could be imposed for estimation purpose. However, it appears difficult to select (statistically or economically) which excess cokurtoses should be set to zero and which not. Keweloh (2020) assumes independent and non-Gaussian shocks to show that these assumptions imply orthogonal, zero cosknewesses and zero excess-kurtoses structural shocks and that these conditions are sufficient to locally identify structural parameters. This allows him to introduce a parsimonious variant of the GMM estimator. These two recent papers are interested by the GMM estimation of the entire SVAR process when all structural parameters are locally identified by higher order moments. Moreover, neither of these two papers develops a testing procedure to verify the identification conditions for the structural shocks.

This paper is organized as follows. Section 2 motivates, from a simple example, the local, statistical identification through the third and fourth unconditional moments. Section 3 derives the rank

condition for the identification of the structural parameters involved in SVAR processes. Section 4 develops a tractable procedure to test whether the identification condition holds, before the estimation of the structural parameters. Section 5 presents an application related to the identification of the structural parameters determining the dynamic responses of output to fiscal shocks. Section 6 concludes.

2. Motivation

This section motivates how local, statistical identification of SVAR processes can be achieved through higher unconditional moments. To do so, we provide a simple example in which the information related to the fourth moments is exploited. Specifically, we consider the following bivariate SVAR process (in innovation form):

$$\nu_{y,t} = \alpha_d \nu_{p,t} + \omega_d \epsilon_{d,t}, \tag{1}$$

$$\nu_{p,t} = \alpha_s \nu_{y,t} + \omega_s \epsilon_{s,t}. \tag{2}$$

Here, $\nu_{y,t}$ and $\nu_{p,t}$ correspond to the reduced-form innovations associated with the logarithms of the quantity and price of a good. The terms $\epsilon_{d,t}$ and $\epsilon_{s,t}$ are the structural demand and supply shocks with the following unconditional scedastic structure: $E[\epsilon_{d,t}^2] = 1$, $E[\epsilon_{s,t}^2] = 1$, and $E[\epsilon_{d,t}\epsilon_{s,t}] = 0$. The parameters α_d and α_s are related to the slopes of the demand and supply curves, whereas the parameters ω_d and ω_s are related to the shifts of the curves following the structural shocks.

System (1)–(2) involves four parameters that have to be identified: α_d, α_s, ω_d, and ω_s. This system can be rewritten as:

$$\begin{pmatrix} \nu_{y,t} \\ \nu_{p,t} \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} \begin{pmatrix} \epsilon_{d,t} \\ \epsilon_{s,t} \end{pmatrix} = \Theta \epsilon_t.$$

As well known, only three of the four structural parameters can potentially be identified through the distinct elements of the unconditional covariance matrix of the reduced-form innovations: $E[\nu_{y,t}^2]$, $E[\nu_{p,t}^2]$, and $E[\nu_{y,t}\nu_{p,t}]$. Importantly, the whole set of parameters could potentially be identified through higher unconditional moments, reflecting, for example, asymmetric and non-mesokurtic distributions of the structural shocks.

As a starting point, consider a parametrization of equations (1)–(2) for which the structural shocks have zero skewnesses and excess kurtoses: $\alpha_d = -.5, \alpha_s = 0.5, \omega_d = \omega_s = 1.0, \epsilon_{d,t} \sim N(0,1)$, and $\epsilon_{s,t} \sim N(0,1)$. Figure 1a) shows the scatter plot of simulated series generated from 10,000 draws for the parametrization with two gaussian shocks. For this case, the simulated reduced-form innovations form a spherical cloud in the ($\nu_{y,t}$, $\nu_{p,t}$) plan. In this context, shifts in the demand and

supply curves are as likely to generate the realizations of $\nu_{y,t}$ and $\nu_{p,t}$. Consider now a rotation of Θ by an orthogonal matrix :

$$Q = \begin{pmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{pmatrix}$$

such that $QQ' = I$. Figure 1b) shows the scatter plot for the same generated shocks but with the orthogonal transformation $\Theta^* = \Theta Q$. The two scatter plots are indistinguishables. Consequently, these two sets of realizations are not informative about the respective slope of the two curves. In this context, possible identification strategies are to impose one short-run or long-run restriction to identify α_s (e.g. Sims, 1980; Blanchard and Quah, 1989). For example, the short-run restriction $\theta_{12} = 0$ implied that the four structural parameters α_d, α_s, ω_d, and ω_s can be recovered.

In contrast, contemplate a parametrization where the shock $\epsilon_{d,t}$ is symmetric, but displays a positive excess kurtosis: $1.291 \times \epsilon_{d,t} \sim t(5)$. The fourth moment of $\epsilon_{d,t}$ translates into a large positive excess kurtosis for $\nu_{y,t}$ and a small excess kurtosis for $\nu_{p,t}$ which generates pronounced leftward and rightward shifts of the demand curve (relative to those associated with the supply curve). These shifts of the demand curve imply movements along the supply curve, so that it becomes possible to identify the slope of the supply, α_s. Figure 1c) shows that this occurs because the extreme realizations of $\epsilon_{d,t}$ compared to those of $\epsilon_{s,t}$. Figure 1d) shows the scatter plot for the same generated shocks but with the orthogonal rotation $\Theta^* = \Theta Q$. Now, the two scatter plots have a different shape such that it becomes possible to identify the slope of the demand and the supply curves. In other words, Θ is unique respective to all orthogonal matrices Q (up to changes in sign and permutations of columns) in presence of excess kurtosis which implies that the structural parameters are locally identified.

It is now well known that all structural parameters are locally, statistically identified when $\epsilon_{d,t}$ is non-normally distributed (see Common, 1994; Eriksson and Koivunen, 2004; Herwartz, 2015; Gouriéroux et al., 2017). Our example suggests that a specific feature of the non-normality, namely the existence of excess kurtosis, is sufficient to ensure the identification of the structural shocks and their generated dynamics. For example, the impact matrix Θ is locally, statistically identified when $\epsilon_{d,t}$ exhibits a time-varying conditional variance (e.g. Normandin and Phaneuf, 2004; Lanne et al., 2010; Lütkepohl and Netšunajev, 2014; Lütkepohl and Schlaak, 2018). This is because conditional heteroskedasticity typically implies positive unconditional excess kurtosis (even for the case of conditional mesokurtic distributions), and, as discussed above, it is precisely the presence of the unconditional non-mesokurtic shock $\epsilon_{d,t}$ that leads to the identification.

Taken altogether, this example suggests that exploiting the information of the structural shocks related to higher unconditional moments help to identify additional parameters of a SVAR process

(relative to the case where only the second unconditional moments are considered).

3. Identification

In this section, we first present the SVAR specification. We then derive the sufficient condition of local, statistical identification through higher unconditional moments.

3.1 Specification

We consider a structural system that takes the form of the following p-order SVAR process:

$$\Phi x_t = \Phi_0 + \sum_{\tau=1}^{p} \Phi_\tau x_{t-\tau} + \epsilon_t. \tag{3}$$

The $(n \times 1)$ vector x_t includes the variables of interest. The $(n \times 1)$ vector ϵ_t contains the structural shocks. These shocks are assumed to display zero cross-sectional unconditional covariances, coskewnesses, and excess cokurtoses. The $(n \times 1)$ vector Φ_0 incorporates n unrestricted intercepts. The non-singular $(n \times n)$ matrix Φ captures n^2 unrestricted contemporaneous interactions among the variables. The $(n \times n)$ matrix Φ_τ contains n^2 unrestricted dynamic feedbacks between the variables.

The first four unconditional moments of the structural shocks of system (3) are obtained from the following expressions:

$$M_\epsilon = E[\epsilon_t], \tag{4}$$

$$\Sigma_\epsilon = E[\epsilon_t \epsilon_t'], \tag{5}$$

$$S_\epsilon = E[\epsilon_t \epsilon_t' \otimes \epsilon_t'], \tag{6}$$

$$K_\epsilon^e = K_\epsilon - K_{\tilde{\epsilon}} = E[\epsilon_t \epsilon_t' \otimes \epsilon_t' \otimes \epsilon_t'] - E[\tilde{\epsilon}_t \tilde{\epsilon}_t' \otimes \tilde{\epsilon}_t' \otimes \tilde{\epsilon}_t'], \tag{7}$$

where E is the unconditional expectation operator, $\otimes$ denotes the Kronecker product and $\tilde{\epsilon}_t$ are hypothetical structural shocks following a multivariate normal distribution. As is common practice, the $(n \times 1)$ vector of expectations is fixed to $M_\epsilon = [\mu_{\epsilon,i}] = 0$ and the $(n \times n)$ covariance matrix is set to $\Sigma_\epsilon = [\sigma_{\epsilon,ij}] = I$ (for $i, j = 1, \ldots, n$), where the latter expression implies that all covariances are assumed to be null, $\sigma_{\epsilon,ij} = 0$ (for $i \neq j$). Also, the $(n \times n^2)$ coskewness matrix concatenates n symmetric $(n \times n)$ submatrices: $S_\epsilon = [S_{\epsilon,1}, \ldots, S_{\epsilon,n}]$, where $S_{\epsilon,k} = [s_{\epsilon,k,ij}] = [E[\epsilon_{k,t}\epsilon_{i,t}\epsilon_{j,t}]]$. The n unconstrained skewnesses of the structural shocks may be non-zero, $s_{\epsilon,k,kk} \neq 0$, whereas all coskewnesses are assumed to be null, $s_{\epsilon,k,ii} = s_{\epsilon,k,ij} = 0$ (for $i, j \neq k$). Finally, the $(n \times n^3)$ excess cokurtosis matrix, K_ϵ^e, is the difference between the cokurtosis matrix, K_ϵ, of the true structural shocks, ϵ_t, and the cokurtosis matrix, $K_{\tilde{\epsilon}}$, associated with

hypothetical structural shocks, $\tilde{\epsilon}_t$, following a multivariate normal distribution. The excess cokurtosis matrix stacks n^2 symmetric $(n \times n)$ submatrices: $K^e_\epsilon = [K^e_{\epsilon,11}, \ldots, K^e_{\epsilon,1n}, \ldots, K^e_{\epsilon,n1}, \ldots, K^e_{\epsilon,nn}]$, where $K^e_{\epsilon,k\ell} = [\kappa^e_{\epsilon,k\ell,ij}] = [E[\epsilon_{k,t}\epsilon_{\ell,t}\epsilon_{i,t}\epsilon_{j,t}] - E[\tilde{\epsilon}_{k,t}\tilde{\epsilon}_{\ell,t}\tilde{\epsilon}_{i,t}\tilde{\epsilon}_{j,t}]]$. The n unconstrained excess kurtoses may be non-zero, $\kappa^e_{\epsilon,kk,kk} \neq 0$, whereas the excess cokurtoses are assumed to be null, $\kappa^e_{\epsilon,kk,ii} = \kappa^e_{\epsilon,kk,ki} = \kappa^e_{\epsilon,kk,ij} = \kappa^e_{\epsilon,k\ell,ij} = 0$.[8]

Next, the reduced form associated with system (3) corresponds to the following p-order VAR process:

$$x_t = \Gamma_0 + \sum_{\tau=1}^{p} \Gamma_\tau x_{t-\tau} + \nu_t, \tag{8}$$

where $\Gamma_0 = \Theta\Phi_0$, $\Gamma_\tau = \Theta\Phi_\tau$, and the non-singular matrix $\Theta = \Phi^{-1}$ captures the impact responses of the variables of interest to the various structural shocks, whereas ν_t includes the reduced-form innovations. These innovations are related to the structural shocks as follows:

$$\nu_t = \Theta\epsilon_t. \tag{9}$$

Also, the first four unconditional moments of the reduced-form innovations are:

$$M_\nu = E[\nu_t], = \Theta M_\epsilon, \tag{10}$$

$$\Sigma_\nu = E[\nu_t\nu_t'] = \Theta\Sigma_\epsilon\Theta', \tag{11}$$

$$S_\nu = E[\nu_t\nu_t' \otimes \nu_t'] = \Theta S_\epsilon(\Theta' \otimes \Theta'), \tag{12}$$

$$K^e_\nu = K_\nu - K_{\tilde{\nu}} = E[\nu_t\nu_t' \otimes \nu_t' \otimes \nu_t'] - E[\tilde{\nu}_t\tilde{\nu}_t' \otimes \tilde{\nu}_t' \otimes \tilde{\nu}_t'] = \Theta K^e_\epsilon(\Theta' \otimes \Theta' \otimes \Theta'). \tag{13}$$

Here, $M_\nu = [\mu_{\nu,i}] = 0$ given that $M_\epsilon = 0$ and $\Sigma_\nu = [\sigma_{\nu,ij}] = \Theta\Theta'$ since $\Sigma_\epsilon = I$. Moreover, $S_\nu = [S_{\nu,1}, \ldots, S_{\nu,n}]$ with $S_{\nu,k} = [s_{\nu,k,ij}] = [E[\nu_{k,t}\nu_{i,t}\nu_{j,t}]]$ and $K^e_\nu = [K^e_{\nu,11}, \ldots, K^e_{\nu,1n}, \ldots, K^e_{\nu,n1}, \ldots, K^e_{\nu,nn}]$ with $K^e_{\nu,k\ell} = [\kappa^e_{\nu,k\ell,ij}] = [E[\nu_{k,t}\nu_{\ell,t}\nu_{i,t}\nu_{j,t}] - E[\tilde{\nu}_{k,t}\tilde{\nu}_{\ell,t}\tilde{\nu}_{i,t}\tilde{\nu}_{j,t}]]$ where ν_t captures the true reduced-form innovations and $\tilde{\nu}_t$ contains hypothetical reduced-form innovations following a multivariate normal distribution. As is well known, the symmetric matrix Σ_ν contains $\frac{n(n+1)}{2}$ distinct elements. Furthermore, the matrices S_ν and K^e_ν include $\frac{n(n+1)(n+2)}{6}$ and $\frac{n(n+1)(n+2)(n+3)}{24}$ distinct elements. The unconditional moments (11), (12) and (13) corresponds respectively to the second-order, third-order and fourth-order cumulants of ν_t.

3.2 Identification

We now determine the conditions for local, statistical identification for the parameters associated with the structural form (3) from the distinct elements associated with the reduced form (8). The

[8] Note that for the cokurtosis matrix $K_{\tilde{\epsilon}}$ associated to a multivariate normal distribution, $\kappa^e_{\tilde{\epsilon},kk,kk} = 3$, $\kappa^e_{\tilde{\epsilon},kk,ii} = 1$, and $\kappa^e_{\tilde{\epsilon},kk,ki} = \kappa^e_{\tilde{\epsilon},kk,ij} = \kappa^e_{\tilde{\epsilon},k\ell,ij} = 0$ (for $\ell, i, j \neq k$).

third and fourth order moments are nonlinear functions of the structural parameters, so verifying identification is more challenging. To ease the exposition, we first consider a specific case that exploits only the excess kurtoses of the structural shocks.[9] Then, we comtemplate a general case that allows for both the skewnesses and excess kurtoses of the structural shocks. For each case, we elaborate the conditions required to identify the impact responses involved in Θ and the skewnesses and/or excess kurtoses of the structural shocks included in S_ϵ and/or K_ϵ^e from the unconditional moments of the reduced-form innovations contained in Σ_ν, S_ν and/or K_ν^e.[10] For completeness, note that, once these parameters are identified, it is trivial to identify the other structural parameters included in Φ_0 and Φ_τ (where $\tau = 1, \ldots, p$) through the relations $\Phi_0 = \Theta^{-1}\Gamma_0$ and $\Phi_\tau = \Theta^{-1}\Gamma_\tau$.

We denote by η and ρ the number of parameters involved in the structural form and the number of distinct elements in the reduced form. For illustration purposes, we begin by examining our first case which exploits only the excess kurtoses of the structural shocks. On the one hand, the number of parameters in the structural form is $\eta = n^2 + m_\kappa$, given that there are n^2 and m_κ parameters to identify in the impact response and excess kurtosis matrices, Θ and K_ϵ^e — where m_κ is the number of non-mesokurtic structural shocks. On the other hand, from relations (11) and (13): there are $\rho = \frac{n(n+1)}{2} + \frac{n(n+1)(n+2)(n+3)}{24}$ independent equations to identify the structural parameters.

Consider the partition of the relation (9) as:

$$\nu_t = \begin{pmatrix} \Theta_\kappa & \Theta_{n\kappa} \end{pmatrix} \begin{pmatrix} \epsilon_{\kappa,t} \\ \epsilon_{n\kappa,t} \end{pmatrix}, \tag{14}$$

where ν_t is a vector of the n reduced form innovations while $\epsilon_{\kappa,t}$ and $\epsilon_{n\kappa,t}$ contain the m_κ and $(n - m_\kappa)$ non-mesokurtic and mesokurtic structural shocks. Intuitively, the information contained in K_ν^e contributes to identify the parameters in Θ_κ and K_ϵ^e, whereas Σ_ν contains specific information to identify the parameters in $\Theta_{n\kappa}$. In particular, consider the configuration where $n = 2$ and $m_\kappa = 1$ (where $\epsilon_{\kappa,t} = \epsilon_{1,t}$ is non-mesokurtic). In this case, the five distinct elements involved in K_ν^e — which correspond to $\kappa_{\nu,11,11}^e = \theta_{11}^4 \kappa_{\epsilon,11,11}^e$, $\kappa_{\nu,11,12}^e = \theta_{11}^3 \theta_{21} \kappa_{\epsilon,11,11}^e$, $\kappa_{\nu,11,22}^e = \theta_{11}^2 \theta_{21}^2 \kappa_{\epsilon,11,11}^e$, $\kappa_{\nu,12,22}^e = \theta_{11}\theta_{21}^3 \kappa_{\epsilon,11,11}^e$ and $\kappa_{\nu,22,22}^e = \theta_{21}^4 \kappa_{\epsilon,11,11}^e$ — contain information to identify the three structural parameters incorporated in $\Theta_\kappa = (\theta_{11}\ \theta_{21})'$ and K_ϵ^e. Also, the three distinct elements in Σ_ν — which are $\sigma_{\nu,11} = \theta_{11}^2 + \theta_{12}^2$, $\sigma_{\nu,12} = \theta_{11}\theta_{21} + \theta_{12}\theta_{22}$, and $\sigma_{\nu,22} = \theta_{21}^2 + \theta_{22}^2$ — allow the identification of the remaining two parameters $\Theta_{n\kappa} = (\theta_{12}\ \theta_{22})'$.

We next present the general case which takes into account both the skewnesses and excess

[9] The specific case that focuses exclusively on the skewness of the structural shocks is presented in Appendix A.

[10] Under local identification, the matrix Θ is unique up to changes in sign and permutations of columns.

kurtoses of the structural shocks. To do so, the relation (9) is partitioned as follows:

$$\nu_t = \begin{pmatrix} \Theta_{ss} & \Theta_{\kappa\kappa} & \Theta_{s\kappa} & \Theta_{ns\kappa} \end{pmatrix} \begin{pmatrix} \epsilon_{ss,t} \\ \epsilon_{\kappa\kappa,t} \\ \epsilon_{s\kappa,t} \\ \epsilon_{ns\kappa,t} \end{pmatrix}. \tag{15}$$

Here, the subvectors $\epsilon_{ss,t}$, $\epsilon_{\kappa\kappa,t}$, $\epsilon_{s\kappa,t}$, and $\epsilon_{ns\kappa,t}$ contain, respectively, the m_{ss}, $m_{\kappa\kappa}$, $m_{s\kappa}$, and $(m - m_{ss} - m_{\kappa\kappa} - m_{s\kappa})$ structural shocks that are exclusively skewed, only non-mesokurtic, both asymmetric and non-mesokurtic, and both symmetric and mesokurtic. The numbers of skewed and non-mesokurtic structural shocks are $m_s = m_{ss} + m_{s\kappa}$ and $m_\kappa = m_{\kappa\kappa} + m_{s\kappa}$. In this environment, $\eta = n^2 + [m_s + m_\kappa]$ and there are $\rho = \left[\frac{n(n+1)}{2}\right] + \left[\frac{n(n+1)(n+2)}{6}\right] + \left[\frac{n(n+1)(n+2)(n+3)}{24}\right]$ independent equations involved by (11), (12) and (13) to identify these parameters. Intuitively, S_ν and K^e_ν contribute to identify the parameters in Θ_{ss}, $\Theta_{\kappa\kappa}$, $\Theta_{s\kappa}$, S_ϵ, and K^e_ϵ, whereas Σ_ν contains specific information to identify the parameters in $\Theta_{ns\kappa}$. It is important to note that the number of reduced form innovations that are only skewed, only non-mesokurtic, both asymmetric and non-mesokurtic, or both symmetric and mesokurtic depends on the matrix Θ. For instance, a reduced form innovation ν_{it} could be individually symmetric (but to be co-skewed with one or more reduced form innovations) while being function of two asymmetric structural shocks. At the opposite, the whole vector of reduced form innovations could be asymmetric while only one structural shock is skewed ($m_{ss} = 1$ and $m_{s\kappa} = 0$). For this reason, the empirical validation of the local identification condition cannot be based on testing procedures applied to individual reduced form innovations ν_{it}. We develop hereafter a testing procedure to verify whether the identification condition holds.

3.2.1 Rank Condition

In this section, we formally derive the rank condition and simple formulas which allow practitioners to evaluate easily this sufficient condition. The rank condition $r = \eta$ represents the sufficient condition for the local, statistical identification of the entire structural system, where r corresponds to the rank associated with the unconditional moment matrices of the reduced-form innovations.[11] Extending the developments of Lütkepohl (2007, Chapter 9), we derive this condition from the ranks of the Jacobian matrices associated with the structural parameters to identify.

If it turns out that the entire structural system is not identified according to the sufficient condition, then our approach further allows to establish which structural parameters are identified and which are not. This gives rise to two important implications. First, it permits to assess which

[11] For nonlinear system of equations, a sufficient condition for the local identification is that the Jocabian matrix is of full colunm rank (see Dufour and Hsiao, 2008) but local identification is still possible using higher order derivatives (see Sargan, 1983 and Donovon and Hall, 2018).

structural subsystem is identified. This subsystem generates the effects induced by the asymmetric and/or non-mesokurtic structural shocks. Second, it enables to determine the structural parameters for which some restrictions must be placed on in order to achieve the identification of the entire system. This is required to recover the effects of the symmetric and mesokurtic structural shocks. As far as we know, these key implications have never been examined in previous studies.

Again, we first consider the case which exploits only the excess kurtoses of the structural shocks. As explained above, the number of parameters involved in the structural form is $\eta = n^2 + m_\kappa$. Also, the rank associated with the reduced form is equal to the rank of the following Jacobian matrix:

$$J = \begin{bmatrix} J_{\theta_\kappa} & J_{\theta_{n\kappa}} & J_{\kappa^e_\epsilon} \end{bmatrix} = \begin{bmatrix} J_{\sigma_\nu,\theta_\kappa} & J_{\sigma_\nu,\theta_{n\kappa}} & J_{\sigma_\nu,\kappa^e_\epsilon} \\ J_{\kappa^e_\nu,\theta_\kappa} & J_{\kappa^e_\nu,\theta_{n\kappa}} & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}. \tag{16}$$

Here, $J_{\theta_\kappa} = \begin{bmatrix} J'_{\sigma_\nu,\theta_\kappa} & J'_{\kappa^e_\nu,\theta_\kappa} \end{bmatrix}'$, $J_{\theta_{n\kappa}} = \begin{bmatrix} J'_{\sigma_\nu,\theta_{n\kappa}} & J'_{\kappa^e_\nu,\theta_{n\kappa}} \end{bmatrix}'$, $J_{\kappa^e_\epsilon} = \begin{bmatrix} J'_{\sigma_\nu,\kappa^e_\epsilon} & J'_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}'$, and $J_{y,x} = \frac{\partial y}{\partial x'}$. Moreover, the vector σ_ν vectorizes the lower triangular part of the symmetric covariance matrix Σ_ν, and the vector κ^e_ν collects the distinct elements of the excess cokurtosis matrix K^e_ν. Finally, the vector θ_κ stacks the columns of the matrix Θ_κ in system (14), the vector $\theta_{n\kappa}$ contains the elements of the matrix $\Theta_{n\kappa}$ and the vector κ^e_ϵ includes the non-zero elements of the excess kurtosis matrix K^e_ϵ.

The rank of the Jacobian matrix (16), $r = rk[J]$, can be evaluated from the analytical derivatives.[12] From these derivatives, we deduce simple formulas to evaluate the rank r, which can be easily assessed from the number of variables involved in the system, n, and the number of non-mesokurtic structural shocks, m_κ. Specifically, the rank corresponds to the sum of three components: $r = r_\kappa + r_{n\kappa} + r_{\kappa^e_\epsilon}$, with $r_\kappa = rk[J_{\theta_\kappa}] = n \times m_\kappa$, $r_{n\kappa} = rk[J_{\theta_{n\kappa}}] = \frac{n(n+1)}{2} - \frac{m_\kappa(m_\kappa+1)}{2}$, and $r_{\kappa^e_\epsilon} = rk\left[J_{\kappa^e_\epsilon}\right] = m_\kappa$. We show in Appendix C that the rank of the matrix J is equal to the sum $r_\kappa + r_{n\kappa} + r_{\kappa^e_\epsilon}$ for all admissible impact matrix Θ.

The components $r_\kappa = n \times m_\kappa$ and $r_{\kappa^e_\epsilon} = m_\kappa$ reveal that the information contained in the second and fourth moments of the reduced-form innovations, Σ_ν and K^e_ν, allows to identify all the $n \times m_\kappa$ elements of the matrix Θ_κ relating the reduced-form innovations to the non-mesokurtic structural shocks, as well as all the m_κ non-zero elements of the excess kurtosis matrix K^e_ϵ. The intuition for this result can be gained from the two following features. First, $rk[J_{\kappa^e_\nu,\theta_\kappa}] = n \times m_\kappa$ and $rk[J_{\kappa^e_\nu,\kappa^e_\epsilon}] = m_\kappa$, but $rk\begin{bmatrix} J_{\kappa^e_\nu,\theta_\kappa} & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix} = n \times m_\kappa$. This implies that the excess cokurtosis matrix K^e_ν identifies the elements of Θ_κ and K^e_ϵ jointly, but not separately. To illustrate this, consider the configuration where $n = 2$ and $m_\kappa = 1$ (where $\epsilon_{\kappa,t} = \epsilon_{1,t}$ is non-mesokurtic), so that $\Theta_\kappa = \begin{pmatrix} \theta_{11} & \theta_{21} \end{pmatrix}'$. In this context, the five distinct elements involved in K^e_ν — which correspond

[12] The derivatives are relegated in Appendix B.

to $\kappa^e_{\nu,11,11} = \theta^4_{11}\kappa^e_{\epsilon,11,11}$ $\kappa^e_{\nu,11,12} = \theta^3_{11}\theta_{21}\kappa^e_{\epsilon,11,11}$, $\kappa^e_{\nu,11,22} = \theta^2_{11}\theta^2_{21}\kappa^e_{\epsilon,11,11}$, $\kappa^e_{\nu,12,22} = \theta_{11}\theta^3_{21}\kappa^e_{\epsilon,11,11}$, and $\kappa^e_{\nu,22,22} = \theta^4_{21}\kappa^e_{\epsilon,11,11}$ — identify the parameters θ_{11}, θ_{21}, and $\kappa^e_{\epsilon,11,11}$ jointly, but not individually. Second, $J_{\sigma_\nu,\theta_\kappa} \neq 0$ whereas $J_{\sigma_\nu,\kappa^e_\epsilon} = 0$. This implies that the covariance matrix Σ_ν disentangles the parameters involved in Θ_κ from those contained in K^e_ϵ, so that it becomes possible to identify individually each parameter in Θ_κ and K^e_ϵ. Coming back to the previous example, the three distinct elements in Σ_ν — which are $\sigma_{\nu,11} = \theta^2_{11} + \theta^2_{12}$, $\sigma_{\nu,12} = \theta_{11}\theta_{21} + \theta_{12}\theta_{22}$, and $\sigma_{\nu,22} = \theta^2_{21} + \theta^2_{22}$ — disentangle the parameters θ_{11} and θ_{21} from $\kappa^e_{\epsilon,11,11}$, given that the variances and covariance are related to θ_{11} and θ_{21} but not to $\kappa^e_{\epsilon,11,11}$.

The component $r_{n\kappa} = \frac{n(n+1)}{2} - \frac{m_\kappa(m_\kappa+1)}{2}$ indicates whether the remaining information contained in the second moments of the reduced-form innovations, Σ_ν, allows to identify all the $n \times (n - m_\kappa)$ elements of the matrix $\Theta_{n\kappa}$ relating the reduced-form innovations to the mesokurtic structural shocks. The intuition for this result is obtained from the following features: $J_{\kappa^e_\nu,\theta_{n\kappa}} = 0$ and $J_{\sigma_\nu,\theta_{n\kappa}} \neq 0$. This implies that only the information captured in Σ_ν, independent of that already used to identify Θ_κ, can be exploited to identify the parameters included in $\Theta_{n\kappa}$. More formally, the covariance matrix of the reduced-form innovations can be rewritten as the sum of $\Sigma_\nu = \Sigma^{m_\kappa}_\nu + \Sigma^{n-m_\kappa}_\nu = \Theta_\kappa\Theta'_\kappa + \Theta_{n\kappa}\Theta'_{n\kappa}$. The matrix $\Sigma^{n-m_\kappa}_\nu$ corresponds to the contribution of the $n - m_\kappa$ mesokurtic structural shocks. This $n \times n$ matrix is symmetric but of rank equals $n - m_\kappa$. This means that $n - m_\kappa$ columns (or rows) are linearly independent. For any of the $n - m_\kappa$ linearly independent columns of the symmetric matrix $\Sigma^{n-m_\kappa}_\nu$, the number of independent elements equals to $\frac{n(n+1)}{2} - \frac{m_\kappa(m_\kappa+1)}{2}$.

Our findings parallel the existing results. These results highlight that, under the more restrictive assumption of independent structural shocks, all the structural parameters are locally, statistically identified when at least all, but one, structural shocks are non-normally distributed (see Comon, 1994; Eriksson and Koivunen, 2004; Herwartz, 2015; Gouriéroux et al., 2017; Keweloh, 2020). Our findings state that the entire structural system is locally, statistically identified when all but one, structural shocks are non-mesokurtic. Specifically, when all structural shocks exhibit excess kurtosis, $m_\kappa = n$, then all the structural parameters are identified as $\eta = r = n^2 + n$ — where $\eta = n^2 + m_\kappa = n^2 + n$ and $r = r_\kappa + r_{n\kappa} + r_{\kappa^e_\epsilon}$, with $r_\kappa = n^2$, $r_{n\kappa} = 0$, and $r_{\kappa^e_\epsilon} = n$. When all, but one, structural shocks are non-mesokurtic, $m_\kappa = n - 1$, then all the structural parameters are identified as $\eta = r = n^2 + n - 1$, where $r_\kappa = n(n-1)$, $r_{n\kappa} = n$, and $r_{\kappa^e_\epsilon} = n - 1$.

Importantly, our approach further provides insights when the entire structural system is not identified. In particular, as already explained above, the moments Σ_ν and K^e_ν allow to locally identify the $n \times m_\kappa$ structural parameters included in Θ_κ and the m_κ distinct elements involved in K^e_ϵ. In general, this means that the structural parameters in m_κ arbitrary columns of Θ are identified.

For example, system (14), $\nu_t = \begin{pmatrix}\Theta_\kappa & \Theta_{n\kappa}\end{pmatrix}\epsilon_t$ with $\epsilon_t = \begin{pmatrix}\epsilon_{\kappa,t} & \epsilon_{n\kappa,t}\end{pmatrix}'$, orders the structural shocks in a way that the parameters in the first m_κ columns are identified. Alternatively, the system $\nu_t = \begin{pmatrix}\Theta_{n\kappa} & \Theta_\kappa\end{pmatrix}\epsilon_t^*$ with $\epsilon_t^* = \begin{pmatrix}\epsilon_{n\kappa,t} & \epsilon_{\kappa,t}\end{pmatrix}'$ changes the ordering of the structural shocks, such that the parameters in the last m_κ columns are identified. For a given ordering of the structural shocks, the subsystem relating all the reduced-form innovations to the non-mesokurtic structural shocks is identified.[13] This subsystem traces the effects generated by the structural shocks displaying non-zero excess kurtoses. The intuition for this result can be gained from the two following features. First, $rk[J_{\kappa_\nu^e,\theta_\kappa}] = n \times m_\kappa$ and $rk[J_{\kappa_\nu^e,\kappa_\epsilon^e}] = m_\kappa$, but $rk\begin{bmatrix}J_{\kappa_\nu^e,\theta_\kappa} & J_{\kappa_\nu^e,\kappa_\epsilon^e}\end{bmatrix} = n \times m_\kappa$. This implies that the excess cokurtosis matrix K_ν^e identifies the elements of Θ_κ and K_ϵ^e jointly, but not separately. However, by considering the information in the covariance matrix Σ_ν, $rk\begin{bmatrix}J_{\theta_\kappa} & J_{\kappa_\epsilon^e}\end{bmatrix} = n \times m_\kappa + m_\kappa$ which is a sufficient condition to identify all structural parameters included in Θ_κ and the m_κ distinct elements involved in K_ϵ^e. This comes from the fact that the derivatives of upper submatrix $J_{\sigma_\nu,\theta_\kappa}$ of J_{θ_κ} depend on the expression $\Sigma_\nu^{m_\kappa}$ defined above which is only a function of Θ_κ allowing to disentangle the elements of Θ_κ from those of K_ϵ^e. Moreover, it can be shown that the matrix $\begin{bmatrix}J_{\theta_\kappa} & J_{\kappa_\epsilon^e}\end{bmatrix}$ does not depend on the unidentified parameters $\Theta_{n\kappa}$, so the identification of the elements of Θ_κ and K_ϵ^e is invariant to the unidentified elements of the matrix $\Theta_{n\kappa}$. In the next section, we present a simple method to estimate the matrix Θ_κ based on the singular value decomposition of the coskewness and/or excess cokurtosis matrices of the reduced-form innovations.

The under-identification of the entire structural system occurs when the moments Σ_ν do not permit to identify all the $n \times (n - m_\kappa)$ elements contained in $\Theta_{n\kappa}$. As a result, certain restrictions on these structural parameters must be imposed. For illustration purposes, consider the following (linear) short-run restrictions $R\theta_{n\kappa} = q$. In this context, the rank condition holds when:

$$rk[J^+] = rk\begin{bmatrix}J_{\theta_\kappa}^+ & J_{\theta_{n\kappa}}^+ & J_{\kappa_\epsilon^e}^+\end{bmatrix} = rk\begin{bmatrix}J_{\sigma_\nu,\theta_\kappa} & J_{\sigma_\nu,\theta_{n\kappa}} & J_{\sigma_\nu,\kappa_\epsilon^e} \\ J_{\kappa_\nu^e,\theta_\kappa} & J_{\kappa_\nu^e,\theta_{n\kappa}} & J_{\kappa_\nu^e,\kappa_\epsilon^e} \\ 0 & R & 0\end{bmatrix} = \eta, \tag{17}$$

where J^+ is the augmented Jacobian matrix, $J_{\theta_\kappa}^+ = \begin{bmatrix}J'_{\sigma_\nu,\theta_\kappa} & J'_{\kappa_\nu^e,\theta_\kappa} & 0'\end{bmatrix}'$, $J_{\theta_{n\kappa}}^+ = \begin{bmatrix}J'_{\sigma_\nu,\theta_{n\kappa}} & J'_{\kappa_\nu^e,\theta_{n\kappa}} & R'\end{bmatrix}'$, and $J_{\kappa_\epsilon^e}^+ = \begin{bmatrix}J'_{\sigma_\nu,\kappa_\epsilon^e} & J'_{\kappa_\nu^e,\kappa_\epsilon^e} & 0'\end{bmatrix}'$. The rank condition (17) states that $(\eta - r)$ linearly independent restrictions on $\theta_{n\kappa}$ are needed to ensure the local identification of the entire structural system. Hence, if the structural shocks of interest are mesokurtic, then their effects can only be gauged when $(\eta - r)$ restrictions are placed on $\theta_{n\kappa}$. In expression (17), the short-run restrictions imply $(\eta - r)$ constraints on the impact responses of the variables to the mesokurtic structural shocks.

[13] In practice, the ordering is implicitly determined by the analyst's selection of the parameters $\kappa_{\epsilon,ii,ii}^e$ (capturing the non-zero excess kurtosis of the structural shocks) to be estimated. For instance, if $\kappa_{\epsilon,ii,ii}^e \neq 0$ with $i = 1, \ldots, m_\kappa$ are estimated then the parameters in the first m_κ columns of Θ are identified, whereas if the $\kappa_{\epsilon,ii,ii}^e \neq 0$ with $i = (n - m_\kappa + 1), \ldots, n$ are estimated then the parameters in the last m_κ columns are identified.

It is straightforward to show that relevant long-run restrictions imply $(\eta - r)$ constraints on the dynamic responses (evaluated over an infinite horizon) of the variables to the mesokurtic shocks.

We now establish a proposition providing the rank condition for the identification of the structural parameters for the general case where the structural shocks display skewnesses and/or excess kurtoses.

Proposition 1 *For a full rank matrix Θ given the unconditional moments of the reduced-form innovations, Σ_ν, S_ν, and K^e_ν, a sufficient condition to the system of equations (11)–(13) to have a locally unique solution is*

$$\begin{aligned} rk[J] &= rk\begin{bmatrix} J_{\theta_{ss}} & J_{\theta_{\kappa\kappa}} & J_{\theta_{s\kappa}} & J_{\theta_{ns\kappa}} & J_{s_\epsilon} & J_{\kappa^e_\epsilon} \end{bmatrix} \\ &= rk\begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} & J_{\sigma_\nu,\theta_{ns\kappa}} & 0 & 0 \\ J_{s_\nu,\theta_{ss}} & 0 & J_{s_\nu,\theta_{s\kappa}} & 0 & J_{s_\nu,s_\epsilon} & 0 \\ 0 & J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} & 0 & 0 & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix} = \eta, \end{aligned} \tag{18}$$

where the vector θ_{ss} stacks by columns the $n \times m_{ss}$ parameters involved in the matrix Θ_{ss} of system (15), the vector $\theta_{\kappa\kappa}$ contains the $n \times m_{\kappa\kappa}$ parameters of the matrix $\Theta_{\kappa\kappa}$, the vector $\theta_{s\kappa}$ includes the $n \times m_{s\kappa}$ parameters of the matrix $\Theta_{s\kappa}$, the vector $\theta_{ns\kappa}$ incorporates the $n \times [n - (m_{ss} + m_{\kappa\kappa} + m_{s\kappa})]$ parameters of the matrix $\Theta_{ns\kappa}$, the vector s_ν collects the distinct elements of the coskewness matrix S_ν, the vector s_ϵ includes the non-zero elements of the skewness matrix S_ϵ and $\eta = n^2 + m_s + m_\kappa$ is the number of structural parameters to identify.

Appendix C shows that $rk\,[J] = r$ with $r = r_{ss} + r_{\kappa\kappa} + r_{s\kappa} + r_{ns\kappa} + r_{s_\epsilon} + r_{\kappa^e_\epsilon}$, where $r_{ss} = rk[J_{\theta_{ss}}] = n \times m_{ss}$, $r_{\kappa\kappa} = rk[J_{\theta_{\kappa\kappa}}] = n \times m_{\kappa\kappa}$, $r_{s\kappa} = rk[J_{\theta_{s\kappa}}] = n \times m_{s\kappa}$, $r_{ns\kappa} = rk[J_{\theta_{ns\kappa}}] = \left(\frac{n(n+1)}{2} - \frac{(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})(m_{ss}+m_{\kappa\kappa}+m_{s\kappa}+1)}{2}\right)$, $r_{s_\epsilon} = rk\,[J_{s_\epsilon}] = m_s$, and $r_{\kappa^e_\epsilon} = rk\left[J_{\kappa^e_\epsilon}\right] = m_\kappa$.[14] In this context, **Proposition 1** has three implications. First, the entire structural system is statistically identified up to changes in sign and permutations of columns, that is $\eta = r$, when all but one, structural shocks exhibit non-zero skewnesses and/or excess kurtoses. Second, whether or not $\eta = r$, the subsystem relating all the reduced-form innovations to the asymmetric and/or non-mesokurtic structural shocks is statistically identified up to changes in sign and permutations of columns, given that the information contained in Σ_ν, S_ν, and K^e_ν always allows to recover the structural parameters involved in Θ_{ss}, $\Theta_{\kappa\kappa}$, $\Theta_{s\kappa}, S_\epsilon$, and K^e_ϵ. This result is stated in the following corollary.

Corollary 1 *For a full rank matrix Θ given the unconditional moments of the reduced-form innovations, Σ_ν, S_ν, and K^e_ν,*

$$rk\begin{bmatrix} J_{\theta_{ss}} & J_{\theta_{\kappa\kappa}} & J_{\theta_{s\kappa}} & J_{s_\epsilon} & J_{\kappa^e_\epsilon} \end{bmatrix} = [r_{ss} + r_{\kappa\kappa} + r_{s\kappa}] + [r_{s_\epsilon} + r_{\kappa^e_\epsilon}] \tag{19}$$

[14] The analytical derivatives involved in the Jacobian matrix are detailed in Appendix B.

and $[r_{ss} + r_{\kappa\kappa} + r_{s\kappa}] + [r_{s_\epsilon} + r_{\kappa^e_\epsilon}] = [n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa}] + [m_s + m_\kappa]$ *the numbers of structural parameters involved in* Θ_{ss}, $\Theta_{\kappa\kappa}$, $\Theta_{s\kappa}, S_\epsilon$, *and* K^e_ϵ. *Consequently, the subsystem relating all the reduced-form innovations to the asymmetric and/or non-mesokurtic structural shocks is statistically identified up to changes in sign and permutations of columns. Moreover, the matrices of derivatives in (19) do not depend on the unidentified structural parameters in* $\Theta_{ns\kappa}$.

This result follows directly from Appendix C. **Corollary 1** implies that the structural parameters involved in Θ_{ss}, $\Theta_{\kappa\kappa}$, $\Theta_{s\kappa}, S_\epsilon$, and K^e_ϵ can be estimated even if the parameters in the matrix $\Theta_{ns\kappa}$ are unidentified and the effects induced by the asymmetric and/or non-mesokurtic structural shocks can be recovered.

When some restrictions are placed on the structural parameters ($R \neq 0$), these restrictions are required if the remaining information captured in Σ_ν does not allow to identify all the structural parameters contained in $\Theta_{ns\kappa}$ — that is $r_{ns\kappa} < n \times [n - (m_{ss} + m_{\kappa\kappa} + m_{s\kappa})]$. In this environment, the next corollary states that the entire structural system becomes locally, statistically identified under short-run restrictions if $(\eta - r)$ linearly independent restrictions are imposed on $\Theta_{ns\kappa}$.

Corollary 2 *For a full rank matrix* Θ *given the unconditional moments of the reduced-form innovations,* Σ_ν, S_ν, *and* K^e_ν, *a sufficient condition to the system of equations (11)–(13) to have a locally unique solution is*

$$rk[J^+] \quad = \quad rk \begin{bmatrix} J_{\theta_{ss}} & J_{\theta_{\kappa\kappa}} & J_{\theta_{s\kappa}} & J_{\theta_{ns\kappa}} & J_{s_\epsilon} & J_{\kappa^e_\epsilon} \\ 0 & 0 & 0 & R & 0 & 0 \end{bmatrix} = \eta$$

where the matrix R *forms the short-run restrictions* $R\theta_{ns\kappa} = q$.

Overall, these results reveal that the rank condition can be readily evaluated from the number of variables involved in the system, n, and the numbers of asymmetric and/or non-mesokurtic structural shocks, m_{ss}, $m_{\kappa\kappa}$, and $m_{s\kappa}$.

4. Testing Procedure

In this section, we elaborate a testing procedure to verify the symmetry and excess kurtosis of the structural shocks, prior to the estimation of the SVAR process. Specifically, we develop a tractable procedure to verify whether the rank condition hold by assessing the numbers of asymmetric and/or non-mesokurtic structural shocks. We then outline a bootstrap procedure to improve the small-sample properties of rank tests designed to verify the numbers of structural shocks displaying non-zero skewnesses and/or excess kurtoses.

4.1 Verification of the Rank Condition

As explained above, the rank condition can be verified from the numbers of asymmetric and/or non-mesokurtic structural shocks. However, the structural shocks become measurable only once the SVAR is estimated.[15] To circumvent this problem, we develop a method to test the number of asymmetric and/or non-mesokurtic structural shocks, which relies exclusively on the reduced-form innovations — where the latter can be evaluated from the reduced form (8) before the estimation of the structural form (3). Specifically, the number of skewed structural shocks, m_s, corresponds to the rank of the coskewness matrix of the reduced-form innovations, S_ν. To see this, note that expression (12) implies that $rk[S_\nu] = rk[S_\epsilon]$ given that Θ is a non-singular matrix and, as a result, $(\Theta' \otimes \Theta')$ is a full-rank matrix. Also, $rk[S_\epsilon] = m_s$ because the assumption of zero cross-sectional coskewnesses of the structural shocks implies that the quadratic form of the corresponding skewness matrix is $S_\epsilon S_\epsilon' = diag\left(s^2_{\epsilon,1,11} \quad \cdots \quad s^2_{\epsilon,n,nn}\right)$, and $s^2_{\epsilon,i,ii} \neq 0$ only for $i = 1, \ldots, m_s$ when m_s structural shocks are skewed.

Analogously, the number of non-mesokurtic structural shocks, m_κ, is given by the rank of the excess cokurtosis matrix of the reduced-form innovations, K^e_ν. That is, equation (13) implies that $rk[K^e_\nu] = rk[K^e_\epsilon]$ given that Θ is a non-singular matrix. Also, $rk[K^e_\epsilon] = m_\kappa$ since the assumption of zero cross-sectional excess cokurtoses of the structural shocks leads to $K^e_\epsilon K^{e\prime}_\epsilon = diag\left((\kappa^e_{\epsilon,11,11})^2 \quad \cdots \quad (\kappa^e_{\epsilon,nn,nn})^2\right)$, and $(\kappa^e_{\epsilon,ii,ii})^2 \neq 0$ only for $i = 1, \ldots, m_\kappa$.

Based on the arguments developed above, we present a proposition to determine the number of structural shocks displaying either non-zero skewnesses, excess kurtoses, or both.

Proposition 2 *Given the unconditional third and fourth moments of the reduced-form innovations, S_ν and K^e_ν, the full rank of the impact matrix Θ and the assumption of zero cross-sectional coskewnesses and excess cokurtoses of the structural shocks imply that the number of asymmetric and/or non-mesokurtic structural shocks, $m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$, is equal to the rank of the matrix $\Psi_\nu = \left(S_\nu \quad K^e_\nu\right)$.*

Proposition 2 is obtained as follows. First, equations (12) and (13) are used to highlight that $rk[\Psi_\nu] = rk[\Psi_\epsilon]$ with $\Psi_\nu = \left(\Theta S_\epsilon(\Theta' \otimes \Theta') \quad \Theta K^e_\epsilon(\Theta' \otimes \Theta' \otimes \Theta')\right)$ and $\Psi_\epsilon = \left(S_\epsilon \quad K^e_\epsilon\right)$, given

[15] Empirically, asymmetric (either postive or negative skewness) and leptokurtic behaviors have been extensively documented for stock and bond returns as well as for exchange rates and commodity prices (see for example, Clark, 1973; Boothe and Glassman, 1987; Bekaert and Harvey, 1997; Fujiwara et al., 2013). Likewise, positive excess kurtosis have been detected for several macroeconomic series, including indicators related to the economic activity — e.g. real GDP, the components of the real aggregate expenditure, industrial production, and unemployment — as well as a variety of indices of the cost of living — e.g. GDP deflator and CPI (see for example, Blanchard and Watson, 1986; Kilian, 1998; Bai and Ng, 2005; Lanne et al., 2017; Gouriéroux et al., 2017; Lanne and Luoto, 2019 and Keweloh, 2020). Note that the studies just reported highlight the existence of skewness and/or excess kurtosis for the variables of interest or for the reduced-form innovations related to these variables, but never for the structural shocks.

that Θ is a non-singular matrix. Then, $rk[\Psi_\epsilon] = m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$ because the assumption of zero cross-sectional coskewnesses and excess cokurtoses of the structural shocks leads to $\Psi_\epsilon \Psi_\epsilon' = diag\left(s^2_{\epsilon,1,11} + (\kappa^e_{\epsilon,11,11})^2 \quad \cdots \quad s^2_{\epsilon,n,nn} + (\kappa^e_{\epsilon,nn,nn})^2\right)$, and $s^2_{\epsilon,i,ii} + (\kappa^e_{\epsilon,ii,ii})^2 \neq 0$ for the m_{ss} structural shocks displaying exclusively non-zero skewnesses, the $m_{\kappa\kappa}$ shocks exhibiting only non-zero excess kurtoses, and the $m_{s\kappa}$ shocks featuring both non-zero shewnesses and excess kurtoses.

In summary, the ranks of S_ν, K^e_ν, and Ψ_ν allow to determine m_s, m_κ, and $m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$ before the estimation of the structural form (3). Then, the numbers of structural shocks displaying exclusively non-zero skewnesses, m_{ss}, excess kurtoses, $m_{\kappa\kappa}$, and both, $m_{s\kappa}$, are readily deduced — given that $m_s = m_{ss} + m_{s\kappa}$ and $m_\kappa = m_{\kappa\kappa} + m_{s\kappa}$.[16]

4.2 Bootstrap Procedure

The objective is to develop a strategy to test the rank of the coskewness and excess cokurtosis matrices. Testing the rank of S_ν and K^e_ν involves the computation of variance-covariance matrix of the null space of theses matrices which is not of full rank, an assumption required in most of rank tests.[17,18] For these reason, we proceed with the rank test proposed by Robin and Smith (2000) which does not require this assumption.

Let us define the estimate of the normalized reduced-form innovations corresponds to $\hat{u}_t = \hat{\Omega}^{-1}\hat{\nu}_t$, where $\hat{\nu}_t$ represents the OLS residuals of the reduced form (8) and $\hat{\Omega}$ is a lower triangular matrix obtained from the Cholesky decompostion of the estimated covariance matrix of the OLS residuals; i.e. $\hat{\Sigma}_\nu = \hat{\Omega}\hat{\Omega}'$. The rank test to determine m_s, m_κ, or $m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$ uses the following likelihood-ratio (LR) and Wald (W) statistics:[19]

$$\widehat{CRT}^{LR}_{r^*} = (T-p)\sum_{i=r^*+1}^{n} \ln(1+\hat{\lambda}_i^2), \tag{20}$$

$$\widehat{CRT}^{W}_{r^*} = (T-p)\sum_{i=r^*+1}^{n} \hat{\lambda}_i^2, \tag{21}$$

where $\hat{\lambda}_i$ are the estimates of the singular values of the matrix S_u, K^e_u, or Ψ_u (with $\hat{\lambda}_1 \geq \ldots \geq$

[16] Specifically, $m_{s\kappa}$ is determined from $m_{ss} + m_{\kappa\kappa} + m_{s\kappa} = (m_s - m_{s\kappa}) + (m_\kappa - m_{s\kappa}) + m_{s\kappa}$, where $m_{ss} + m_{\kappa\kappa} + m_{s\kappa} = rk[\Psi_\nu]$, $m_s = rk[S_\nu]$, and $m_\kappa = rk[K^e_\nu]$. Then, m_{ss} and $m_{\kappa\kappa}$ are determined from $m_{ss} = m_s - m_{s\kappa}$ and $m_{\kappa\kappa} = m_\kappa - m_{s\kappa}$.

[17] The null space of a $m \times n$ matrix A is the set of all vectors x such that $\mathcal{N}(A) = \{x \in \mathbb{R}^n | Ax = 0\}$.

[18] See for example Kleibergen and Paap (2006, Assumption 2) and the review of existing rank tests by Camba-Méndez and Kapetanios (2008). See also Lewis (2019) for the implementation of a rank test to verify identification conditions in the context of SVAR with time-varying volatility.

[19] See Anderson (1951) for the LR form and as well as Bura and Yang (2011) and Portier and Delyon (2014) for the Wald form.

$\hat{\lambda}_n \geq 0$) and r^* is the rank of this matrix under the null hypothesis.[20] The matrices S_u, K_u^e, and Ψ_u are constructed from the sample estimates of the coskweness $\hat{s}_{u,k,ij} = \frac{1}{T-p}\sum_{t=p+1}^{T} \hat{u}_{k,t}\hat{u}_{i,t}\hat{u}_{j,t}$ and cokurtosis $\hat{\kappa}_{u,k\ell,ij} = \frac{1}{T-p}\sum_{t=p+1}^{T} \hat{u}_{k,t}\hat{u}_{\ell,t}\hat{u}_{i,t}\hat{u}_{j,t}$ of the estimated normalized reduced-form innovations, as well as the cokurtoses $\kappa_{\tilde{u},kk,kk} = 3$, $\kappa_{\tilde{u},kk,ii} = \sigma_{\tilde{u},kk}\sigma_{\tilde{u},ii} = 1$ (for $i \neq k$), and $\kappa_{\tilde{u},kk,ki} = \kappa_{\tilde{u},kk,ij} = \kappa_{\tilde{u},k\ell,ij} = 0$ (for $\ell, i, j \neq k$) of hypothetical normal reduced-form innovations.[21] Robin and Smith (2000) show that, under some regularity conditions, the statistics (20) and (21) have limiting distributions that are weighted sums of independent chi-squared variables, despite that the estimators of $vec(S_u)$, $vec(K_u^e)$, and $vec(\Psi_u)$ have not full rank asymptotic covariance matrices. The main drawback of such test is that the statistic (20) and (21) are not pivotal, i.e. their asymptotic distribution depends on the unknown quantities S_u, K_u^e, or Ψ_u and their respective asymptotic variance-covariance matrix. An estimator of the weights of the sum of the independent chi-square distribution can be obtained using consistent estimators of these unknown quantities. This allows to provide an estimation of the asymptotic critical values for the statistics $CRT_{r^*}^{LR}$ and $CRT_{r^*}^{W}$ under the null hypothesis that the rank is r^*. Appendix D shows the derivation of the limiting distribution of the statistics (20) and (21) and how to obtain an estimator of this limiting distribution.

From analytical approximations of the first four moments, it can be shown that $\hat{s}_{u,i,ii}$ has a symmetric leptokurtic distribution which fairly rapidly tends to a normal distribution as the sample size increases, but $\hat{\kappa}_{u,ii,ii}$ has a very skewed distribution that hardly converges to a normal distribution (see Mardia, 1980). This implies that the finite-sample critical values to test the null hypothesis of zero excess kurtosis converge extremely slowly to their asymptotic counterparts. Numerical simulations of the Jarque-Bera tests for kurtosis further suggest that the use of asymptotic critical values leads to severe size distortions, as the empirical size often substantially deviates from the nominal size even for samples as large as $T = 5,000$ (see Kilian and Demiroglu, 2000; Bai and Ng, 2005).

To circumvent this problem, we design a bootstrap procedure to compute the finite-sample critical values for the statistics $CRT_{r^*}^{LR}$ and $CRT_{r^*}^{W}$ associated with the ranks of S_u, K_u^e, and Ψ_u to determine m_s, m_κ, and $m_{ss}+m_{\kappa\kappa}+m_{s\kappa}$. The bootstrap appears here to be a convenient alternative because it avoids the estimation of unknown quantities which are probably imprecisely estimated in finite sample.[22] In particular for K_u^e, this entails evaluating the variance-covariance matrix of

[20] This corresponds to the unweighted case in Robin and Smith (2000), so $\hat{\lambda}_i^2$ are the eigenvalues of the quadratic form of the matrix S_u, K_u^e, or Ψ_u.

[21] Note that $rk[S_u] = rk[S_\nu] = m_s$, $rk[K_u^e] = rk[K_\nu^e] = m_\kappa$, and $rk[\Psi_u] = rk[\Psi_\nu] = m_{ss} + m_{\kappa\kappa} + m_{s\kappa}$.

[22] Since the asymptotic distribution of the statistics is not pivotal, the bootstrap does not provide asymptotic refinements.

the fourth unconditional moments.

The consistency of the bootstrap procedure requires that the bootstrap data satisfy the null hypothesis. To respect this requirement, the procedure here is based on the constrained bootstrap testing procedure of the rank of a matrix proposed by Portier and Delyon (2014). Consider the case of the matrix K_u^e. Suppose that the matrix K_u^e is of rank $r^* \leq n$. The singular value decomposition of K_u^e gives $K_u^e = C\Lambda D'$ where Λ is a diagonal matrix with the singular values on the diagonal. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the singular values ordered in decreasing values. For a matrix K_u^e of rank equal to r^*, the first r singular values are different from zero and the last $n - r^*$ singular values are equal to zero. The orthogonal matrix C contain columns of the associated singular vectors c_i for $i = 1, \ldots, n$ and $CC' = I_n$ whereas the orthogonal matrix D contain columns of the associated eigenvectors d_i for $i = 1, \ldots, n$ and $DD' = I_{n^3}$. This implies that $CC' = C_r C_r' + C_{n-r} C_{n-r}'$ where the submatrix C_r contains the first r columns of C associated with the singular values different from zeros and the submatrix C_{n-r} contains the $n - r$ singular vectors associated with singular values equal to zero. Similarly, $DD' = D_r D_r' + D_{n^3-r} D_{n^3-r}'$ where the submatrix D_r contains the first r columns of D associated with the singular values different from zeros and the submatrix D_{n^3-r} contains the $n^3 - r$ singular vectors associated with singular values equal to zero.

Thus

$$C' K_u^e D = \begin{bmatrix} C_r' K_u^e D_r & C_r' K_u^e D_{n^3-r} \\ C_{n-r}' K_u^e D_r & C_{n-r}' K_u^e D_{n^3-r} \end{bmatrix} = \Lambda \tag{22}$$

The submatrix $C_{n-r}' K_u^e D_{n^3-r}$ corresponds to the null space of K_u^e which is the object of interest (see Al-Sadoon, 2017).

According to Portier and Delyon (2014), statistics (20) and (21) share the following form[23]

$$\sum_{i=r^*+1}^{n} \hat{\lambda}_i^2 = \|vec(\widehat{K}_u^e - \widehat{K}_{u,c}^e)\|^2 = \|vec(\widehat{C}_{n-r^*}' \widehat{K}_u^e \widehat{D}_{n^3-r^*})\|^2 \tag{23}$$

with

$$\widehat{K}_{u,c}^e = \widehat{C}_{r^*} \widehat{\Lambda}_{r^*} \widehat{D}_{r^*}' = \underset{rk(K_u^e)=r^*}{\operatorname{argmin}} \|vec(\widehat{K}_u^e - K_u^e)\|^2$$

where $\widehat{K}_{u,c}^e$ is the constrained excess cokurtosis matrix by the null hypothesis closest to the estimated matrix $\widehat{K}_u^e$ under the Euclidean norm. The matrix $\widehat{K}_{u,c}^e$ is of rank r^* and is given by $\widehat{C}_{r^*} \widehat{\Lambda}_{r^*} \widehat{D}_{r^*}' = \widehat{C}_{r^*} \widehat{C}_{r^*}' \widehat{K}_u^e \widehat{D}_{r^*} \widehat{D}_{r^*}'$ according to the singular value decomposition. By (23) the statistics $CRT_{r^*}^{LR}$ and $CRT_{r^*}^{W}$ are then function of a the null space estimator of the matrix $\widehat{K}_u^e$.

The corresponding bootstrap form is

$$\sum_{i=r^*+1}^{n} \hat{\lambda}_i^{2,b} = \|vec(\widehat{K}_{u,c}^{e,b} - K_{u,c}^{e,b})\|^2 = \|vec(\widehat{C}_{n-r^*}^{b\prime} \widehat{K}_{u,c}^{e,b} \widehat{D}_{n^3-r*}^{b})\|^2$$

[23] The statistic $CRT_{r^*}^{LR} = (T-p) \sum_{i=r^*+1}^{n} \hat{\lambda}_i^2 + o_p(1)$.

with

$$K_{u,c}^{e,b} = \widehat{C}_{r^*}^{b} \widehat{\Lambda}_{r^*}^{b} \widehat{D}_{r^*}^{b\prime} = \underset{rk(K_u^e)=r^*}{\operatorname{argmin}} \|vec(\widehat{K}_{u,c}^{e,b} - K_u^e)\|^2$$

for $b = 1, \ldots, B$ where $\widehat{K}_{u,c}^{e,b}$ is the bootstrapped excess cokurtosis matrix under the null hypothesis with the corresponding singular value decomposition (22) and $K_{u,c}^{e,b}$ is the constrained bootstrapped excess cokurtosis matrix by the null hypothesis closest to $\widehat{K}_{u,c}^{e,b}$ with respect to the Euclidean norm. The question is how to compute $\widehat{K}_{u,c}^{e,b}$ under the null hypothesis. The objective is to bootstrap linear combinations of the normalized reduced-form innovations such that the largest r^* bootstrapped singular values of $\widehat{K}_{u,c}^{e,b}$ mimic the empirical ones with respect to the Euclidean norm and the $r^*+1, \ldots, n$ singular values correspond to the null hypothesis. The singular vectors in the matrix $\widehat{C}_{r^*}$ span the column space corresponding to the largest r^* singular values of $\widehat{K}_u^e$. In Appendix D, we show that the matrix of singular vectors $\widehat{C}_{r^*}$ gives linear combinations of the bootstrapped normalized reduced-form innovations such that the corresponding singular values of $\widehat{K}_{u,c}^{e,b}$ are the closest to the empirical ones. The vector of the bootstrapped normalized reduced-form innovations $u_t^b = \begin{pmatrix} u_{r^*,t}^{b\prime} & u_{n-r^*,t}^{b\prime} \end{pmatrix}'$ is thus generated such that the elements in the subvector $u_{r^*,t}^b$ are obtained by bootstrapping $\widehat{C}_{r^*}'\hat{u}_t$ and the elements in the subvector $u_{n-r^*,t}^b$ are drawned from a symmetric and mesokurtic distribution. This ensures that the largest r^* bootstrapped singular values of $\widehat{K}_{u,c}^{e,b}$ computed with u_t^b mimic the empirical ones with respect to the Euclidean norm and the $r^*+1, \ldots, n$ singular values of $\widehat{K}_{u,c}^{e,b}$ correspond to the null hypothesis. In fact, when r is equal to the true number of structural shocks with excess kurtosis m_κ, the singular vector columns of the matrix $\widehat{C}_r$ corresponding to the r greatest singular values provide a simple estimator of the matrix Θ_κ by $\hat{\Omega}\widehat{C}_\kappa = \widehat{\Theta}_\kappa$. This is deduced from the following relationship between the covariance matrix of the reduced-form innovations and the impact matrix Θ :

$$\Sigma_\nu = \Theta\Theta' = \Omega CC'\Omega' = \Omega C_\kappa C_\kappa'\Omega' + \Omega C_{n\kappa} C_{n\kappa}'\Omega = \Theta_\kappa\Theta_\kappa' + \Theta_{n\kappa}\Theta_{n\kappa}'$$

where $C = [C_\kappa \; C_{n\kappa}]$, $\Theta_\kappa = \Omega C_\kappa$ and Ω is the lower triangular matrix of the Choleski decomposition of Σ_ν. By the normalisation, the estimation of Θ_κ corresponds to the estimation of the orthogonal vector in the matrix C_κ under the contraint that $C_\kappa' C_\kappa = I$.

We now illustrate the various steps of the procedure for the rank of K_u^e.

Step 1. Under the null hypothesis $rk[K_u^e] = r^*$ (i.e. r^* is the assumed number of non-mesokurtic structural shocks), the vector $u_t^b = \begin{pmatrix} u_{r^*,t}^{b\prime} & u_{n-r^*,t}^{b\prime} \end{pmatrix}'$ is generated as follows. The elements contained in the $(r^* \times 1)$ subvector $u_{r^*,t}^b$ are obtained by bootstraping those included in the vector $w_{r^*,t} = \widehat{C}_{r^*}'\hat{u}_t$ for $t = (p+1), \ldots, T$, where $\widehat{C}_{r^*}$ is a $(n \times r^*)$ matrix stacking the left singular vectors associated with the r^* largest singular values of $\widehat{K}_u^e$ and $\hat{u}_t$ is the $(n \times 1)$ vector collecting the estimated normalized reduced-form innovations. This implies that the elements contained in $w_{r^*,t}$ correspond to linear

combinations of the normalized reduced-form innovations which display the largest excess kurtoses. The elements contained in the $[(n-r^*)\times 1]$ subvector $u^b_{n-r^*,t}$ are drawned from the symmetric and mesokurtic distribution $u^b_{n-r^*,t} \sim N(0,I)$ for $t=(p+1),\ldots,T$.

Step 2. The bootstrap sample is generated recursively from the VAR process (8) as:

$$x^b_t = \hat{\Gamma}_0 + \sum_{\tau=1}^{p} \hat{\Gamma}_\tau x^b_{t-\tau} + \hat{\Omega} u^b_t, \tag{24}$$

for $t=(p+1),\ldots,T$. To do so, the starting values of x^b_t for $t=1,\ldots,p$ are generated by randomly drawing a block of the actual data of length p, while $\hat{\Gamma}_0$, $\hat{\Gamma}_\tau$, and $\hat{\Omega}$ are the estimates of the reduced-form parameters obtained by applying OLS on the actual sample. Following Bose (1988), these estimates are treated as the population values of the reduced-form parameters.

Step 3. The VAR process is estimated to yield:

$$x^b_t = \hat{\Gamma}^b_0 + \sum_{\tau=1}^{p} \hat{\Gamma}^b_\tau x^b_{t-\tau} + \hat{\Omega}^b \hat{u}^b_t, \tag{25}$$

where $\hat{\Gamma}^b_0$, $\hat{\Gamma}^b_\tau$, and $\hat{\Omega}^b$ are the estimates obtained by performing OLS on the bootstrap sample, whereas $\hat{u}^b_t$ corresponds to the normalized residuals.

Step 4. The normalized residuals $\hat{u}^b_t$ are used to compute the bootstrap analogues of the statistics (20) and (21).

Step 5. Steps 1 to 4 are repeated for $b=1,\ldots,B$ where $B=1999$ to compute the empirical distributions of the statistics (20) and (21).[24] Selecting the appropriate quantiles of these empirical distibutions yield the finite-sample critical values to test the null hypothesis that the rank is equal to r^* against the alternative hypothesis that the rank is larger than r^*.

Step 6. Steps 1 to 5 are repeated for $r^*=0,1,\ldots,n-1$. If the null hypothesis $rk[K^e_u]=r^*$ is rejected for $r^*=0,1,\ldots,m-1$ but is not rejected for $r^*=m$ with $m<n$, then the number of non-mesokurtic structural shocks corresponds to $m_\kappa=m$. However, if the null hypothesis $rk[K^e_u]=r^*$ is rejected for $r^*=0,1,\ldots,n-1$, then $m_\kappa=n$.

The last step of the bootstrap procedure is similar to the sequential procedure proposed by Robin and Smith (2000). These authors show that, asymptotically, such a sequential procedure never selects a value of r^* that is smaller than the true rank of the matrix of interest.[25]

[24] According to Davidson and Mackinnon (2000) the number of bootstrap replications B must be chosen so that $\alpha(B+1)$ is an integer where α is the chosen level of the test.

[25] In such a sequential procedure there exists a probability, corresponding to the empirical size, to falsely reject the null hypothesis, as is common to usual testing procedures. A false rejection of the null hypothesis implies that the subsequent test assumes that the number of non-mesokurtic structural shocks is $r^*=r+1$ where r is the true number in the data. In this case, the subvector u^b_t corresponds to linear combinations of standardized reduced-form innovations resulting in r non-mesokurtic structural shocks and one mesokurtic structural shock. The subvector

To document the size properties of rank tests for excess kurtosis, we evaluate the empirical sizes by simulating samples of size T form the bivariate system (1)–(2) with $\epsilon_{2,t} \sim N(0,1)$ and $\epsilon_{1,t} \sim N(0,1)$ under the null hypothesis $r^* = 0$ or $1.291 \times \epsilon_{1,t} \sim t(5)$ under $r^* = 1$.[26] Table 1 presents the empirical sizes of the rank tests with asymptotic distributions, where the limiting critical values are computed as in Appendix D. For both the Wald and likelihood-ratio tests, the empirical size is very conservative under the null hypotheses $r^* = 0$ and $r^* = 1$. Specifically, the empirical sizes are systematically close to zero, and, as such, they are substantially smaller than the nominal sizes even for samples as large as $T = 5,000$. Table 1 also reports the empirical sizes related to finite-sample distributions, where the critical values are constructed from the bootstrap procedure developed above. Importantly, the Wald and likelihood-ratio tests are essentially free of size distortions: the empirical sizes are almost identical to the nominal sizes, regardless of the sample size T.

Finally, we report the empirical powers of the tests with finite-sample distributions for the rank of K_u^e. For this purpose, we simulate the bivariate system (1)–(2) for cases in which i) $\epsilon_{2,t}$ is mesokurtic and $\epsilon_{1,t}$ displays non-zero excess kurtosis when we consider the null hypothesis $r^* = 0$, and ii) $\epsilon_{2,t}$ and $\epsilon_{1,t}$ are both non-mesokurtic when we contemplate that $r^* = 1$ — where the excess kurtosis is moderate ($\kappa_\epsilon^e = 1$) or pronounced ($\kappa_\epsilon^e = 6$). Table 2 highlights two main features. First, as expected, the powers of the tests substantially improve as the sample size increases. Second, the powers of the tests considerably increase as the excess kurtosis becomes more pronounced.[27] Accordingly, a small sample size and/or a negligible excess kurtosis lead to a conservative analysis: it is likely that an analyst would conclude that the entire system is under-identified (even if it is actually identified) or would under-evaluate the size of the subsystem that is identified (when the entire system is actually under-identified).

Overall, our bootstrap procedure for rank tests always overcomes size distortions and often yields good power properties. This also holds for the bootstrap procedure applied to the coskewness matrix S_u and the matrix Ψ_u that combines the coskewness and the excess cokurtosis matrices.[28]

$u^b_{n-r^*,t}$ is still drawing from a symmetric and mesokurtic distribution which corresponds to the null hypothesis. Simulation results not reported here show that test based on the null that $r^* = r + 1$ is conservative. However, a false rejection of the null could result in adverse consequences for the subsequent inferences based on the misspecified SVAR. Moreover, even for a sequential procedure providing a consistent estimate of the rank, Leeb and Pötscher (2005) show that the finite-sample distribution for the subsequent inferences may not be well approximated by the pointwise asymptotic. However, this corresponds to the worst possible outcome when conducting inference, not the likely outcome (see Killian and Lütkepohl, 2017, Chapter 2).

[26] The empirical sizes and powers of rank tests for skewness are reported in Appendix E.

[27] The results of the Wald and the LR statistics are the same in Table 2 for $r^* = 1$ because both statistics are small and are almost of the same values.

[28] Simulation results are reported for the coskewness matrix in Appendix E. We also perform simulations for a trivariate system with parameters calibrated according to the application appearing in the next section. The results are really close to the ones with two variables and can be obtained upon request.

Consequently, this procedure is most useful to determine the numbers of asymmetric and/or non-mesokurtic structural shocks, in order to assess whether the rank condition holds.

5. Application

We now apply the developments presented above to document the effects of fiscal policies on economic activity. The effectiveness of fiscal policies represents a classic question in macroeconomics. It has received renewed interest in light of the recent Great Recession and the ongoing debate about which type of government interventions stimulate the economy the most.

We consider a trivariate SVAR process:

$$\begin{pmatrix} \nu_{\tau,t} \\ \nu_{g,t} \\ \nu_{y,t} \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{pmatrix} \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{pmatrix}, \tag{26}$$

where $\nu_{\tau,t}$, $\nu_{g,t}$, $\nu_{y,t}$ represent the reduced-form innovations capturing the unanticipated movements in taxes, government spending, and output, whereas $\epsilon_{1,t}$, $\epsilon_{2,t}$, and $\epsilon_{3,t}$ correspond to the structural shocks.

The relation (26) is evaluated for quarterly U.S. data from 1980-I to 2015-III.[29] Output corresponds to the logarithm of real GDP per capita, taxes are defined as the logarithm of real total government receipts net of transfer payments per capita, and government spending is the logarithm of the sum of real government consumption and gross government investment expenditures per capita. The series are expressed in real terms using the GDP deflator and in per capita terms using total population. Also, taxes and government spending are measured for the general government, i.e. the sum of federal (defense and non-defense), state, and local governments. The data are seasonally adjusted at the source and are taken from the National Income and Products Accounts (NIPA), except for total population which is obtained from the Federal Reserve Bank of Saint-Louis' FRED database. The reduced form (8) includes a linear deterministic trend and eight lags, which correspond to the most parsimonious lag structure for which all reduced-form residuals are serially uncorrelated.

We verify whether the identification condition holds before proceeding to the estimation of the structural parameters. To do so, we apply the rank tests where the finite-sample critical values are computed by the bootstrap procedure discussed in Section 4.2. The results reveal that the hypothesis stipulating that the structural shocks are symmetric is not rejected (at all conventional levels) and only one shock is non-mesokurtic (i.e. $m_{ss} = m_{s\kappa} = 0$ and $m_{\kappa\kappa} = 1$), given that the

[29] A similar starting date of the sample is selected by Perotti (2004), Favero and Giavazzi (2009), and Bouakez et al. (2014).

likelihood-ratio and Wald versions of the tests imply that $rk[S_u] = m_s = 0$, $rk[K_u^e] = m_\kappa = 1$, and $rk[\Psi_u] = m_{ss} + m_{\kappa\kappa} + m_{s\kappa} = 1$. In this context, the number of structural parameters is $\eta = n^2 + m_\kappa = 10$, whereas the number of distinct elements in the reduced form is $\rho = \left[\frac{n(n+1)}{2}\right] + \left[\frac{n(n+1)(n+2)(n+3)}{24}\right] = 21$ and the rank associated with the reduced form is $r = r_\kappa + r_{n\kappa} + r_{\kappa_\epsilon^e} = 9$ — with $r_\kappa = n \times m_\kappa = 3$, $r_{n\kappa} = \frac{n(n+1)}{2} - \frac{m_{\kappa\kappa}(m_{\kappa\kappa}+1)}{2} = 5$, and $r_{\kappa_\epsilon^e} = m_\kappa = 1$. This implies that the rank (sufficient) condition $r = \eta$ is violated so that the entire system is not identified and to achieve the idenfication $(\eta - r) = 1$ restriction must be imposed

We next estimate the structural parameters involved in the subsystem of (26) that is locally, statistically identified. These parameters are arbitrarily selected to be the elements of the first column of Θ (i.e. θ_{11}, θ_{21}, θ_{31}) and $\kappa^e_{\epsilon,11,11}$. The estimation of these parameters is performed by minimizing the following moment matching function:

$$(\hat{\zeta}_\nu - \zeta_\nu(\Theta, \kappa^e_{\epsilon,11,11}))'(\hat{\zeta}_\nu - \zeta_\nu(\Theta, \kappa^e_{\epsilon,11,11})), \tag{27}$$

under $(\eta - r) = 1$ restriction that either $\theta_{23} = 0$ or $\theta_{32} = 0$.. Note that each of these restrictions ensures that the entire system is identified, but it is not placed on the parameters of interest θ_{11}, θ_{21}, θ_{31}, and $\kappa^e_{\epsilon,11,11}$. Also, $\zeta_\nu = \begin{pmatrix}\sigma_\nu & \kappa^e_\nu\end{pmatrix}'$, where σ_ν vectorizes the lower triangular part of the symmetric covariance matrix Σ_ν obtained from expression (11) and κ^e_ν collects all the distinct elements of the excess cokurtosis matrix K^e_ν obtained from (13). The vector $\hat{\zeta}_\nu = \begin{pmatrix}\hat{\sigma}_\nu & \hat{\kappa}^e_\nu\end{pmatrix}'$ includes the sample estimates of all the second and fourth unconditional moments of the reduced-form residuals. As explained previously, the information contained in these moments allows to identify all the structural parameters relating the reduced-form innovations to the non-mesokurtic structural shock, as well as the excess kurtosis of this shock. Finally, the confidence intervals of the estimates are computed from 5000 bootstrap samples. The implemented estimation procedure corresponds to the Generalized Method of Moments (GMM) with a fixed weight matrix (i.e. the identity matrix in our case), which yields estimators that are consistent and asymptotically normal (see Hansen, 1982). Moreover, in this context the validity of the bootstrap procedure is established in Hall and Horowitz (1996). This estimation procedure is consistent, but not optimal, since we do not use the optimal weighting matrix. However, it is not clear in this case that the small-sample behavior of the GMM estimation with the optimal weighting matrix outperforms the GMM estimation with the identity matrix. Specifically, the estimation of the optimal weighting matrix entails evaluating the covariance matrix of the fourth unconditional moments, which tends to be quite imprecisely estimated in small samples (see Bonhomme and Robin, 2009; Keweloh, 2020).

In the estimation procedure, we normalize the estimate of θ_{11} to be positive, so that we consider the case where the impact response of $\nu_{\tau,t}$ to $\epsilon_{1,t}$ is positive. We also identify the elements in the

first column of Θ, given that we have selected $\epsilon_{1,t}$ as being the structural shock that displays excess kurtosis. Table 3 shows that the estimates of θ_{11} and $\kappa^e_{\epsilon,11,11}$ are numerically sizable and statistically significant, whereas the estimates of θ_{21} and θ_{31} are negligible and insignificant. These results hold whether the restriction $\theta_{23} = 0$ or $\theta_{32} = 0$ is invoked. This occurs because the parameters θ_{11}, θ_{21}, θ_{31}, and $\kappa^e_{\epsilon,11,11}$ are identified, regardless of the restriction imposed on the other parameters of system (26). Given that the null hypotheses $\theta_{21} = 0$ and $\theta_{31} = 0$ are not rejected, this suggests that the term $\nu_{\tau,t}$ exhibits non-zero excess kurtosis, while $\nu_{g,t}$ and $\nu_{y,t}$ display zero excess kurtoses and excess cokurtoses.[30] Also, the values $\theta_{21} = 0$ and $\theta_{31} = 0$ imply that, at impact, the structural shock $\epsilon_{1,t}$ only affects taxes, so that this shock can be interpreted economically as a tax shock, i.e. $\epsilon_{\tau,t} = \epsilon_{1,t}$. In this specific application, the statistical properties of the subsystem linking the reduced-form innovations to the non-mesokurtic structural shock lead to the economic identification of the tax shock. In general, however, local, statistical identification does not guarantee that the structural shocks have an economic interpretation.

The results lead to the important implication that the subsytem relating all the reduced-form innovations to the tax shock is identified. From this subsystem, we find that the effectiveness of the tax policy is weak. That is, the dynamic response of output after a tax shock is small and not statistically significant (see Figure 2), and the tax multiplier (i.e. the dollar change in output occurring in quarter $t+i$ resulting from a dollar cut in the exogenous component of taxes) is small: it is null at impact and it reaches a peak of about 0.61 at 14 quarters (see Table 3). Again, the dynamic response and the tax multiplier are not affected by the selection of the restriction $\theta_{23} = 0$ or $\theta_{32} = 0$, given that the responses of output following a tax shock is not affected by the restriction imposed on the other parameters of system (26).

In contrast, the subsytem relating the reduced-form innovations to the structural shocks $\epsilon_{2,t}$ and $\epsilon_{3,t}$ is under-identified. To achieve the identification of this subsystem, $(\eta - r) = 1$ restriction must be imposed. This restriction is required to assess the responses of output, taxes, and government spending following the structural shocks $\epsilon_{2,t}$ and $\epsilon_{3,t}$, where one of these shocks may correspond to the government spending shock.

To deepen the analysis of the effectiveness of the spending policy, we rely on the economic

[30] These results are corroborated by applying Jarque-Bera tests for the reduced-form innovations, where the finite-sample critical values are approximated by Kilian and Demiroglu's (2000) bootstrap procedure. Specifically, we find that the hypothesis of symmetry is not rejected for all reduced-form innovations, whereas the hypothesis of zero excess kurtosis is rejected only for the reduced-form innovation associated with taxes, $\nu_{\tau,t}$.

specification invoked in the seminal paper of Blanchard and Perotti (2002):

$$\nu_{\tau,t} = \alpha_1 \nu_{y,t} + \alpha_2 \omega_g \epsilon_{g,t} + \omega_\tau \epsilon_{\tau,t}, \tag{28}$$

$$\nu_{g,t} = \beta_1 \nu_{y,t} + \beta_2 \omega_\tau \epsilon_{\tau,t} + \omega_g \epsilon_{g,t}, \tag{29}$$

$$\nu_{y,t} = \gamma_1 \nu_{\tau,t} + \gamma_2 \nu_{g,t} + \omega_y \epsilon_{y,t}. \tag{30}$$

The structural shocks $\epsilon_{\tau,t}$ and $\epsilon_{g,t}$ represent the tax and spending shocks that reflect unexpected, exogenous, discretionary changes in taxes and government expenditures, whereas $\epsilon_{y,t}$ captures the non-fiscal shocks that affect output. Equations (28) and (29) describe the government's tax and spending rules. Specifically, the rule (28) highlights that taxes may vary in response to changes in output or to spending shocks. The rule (29) has an analogous interpretation for public spending. In these rules, the parameters α_1 and β_1 potentially measure the automatic and government's systematic responses of taxes and government spending to changes in output, whereas α_2 and β_2 allow for interactions between tax and spending policies. Equation (30) relates changes in output to changes in taxes and government expenditures, and to non-fiscal shocks. Finally, the terms ω_τ, ω_g, and ω_y are scaling parameters.

The specification (28)–(30) can be expressed in the form of relation (26) as:

$$\begin{pmatrix} \nu_{\tau,t} \\ \nu_{g,t} \\ \nu_{y,t} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} (1+\alpha_1\beta_2\gamma_2 - \beta_1\gamma_2)\omega_\tau & (\alpha_2 + \alpha_1\gamma_2 - \alpha_2\beta_1\gamma_2)\omega_g & \alpha_1\omega_y \\ (\beta_2 + \beta_1\gamma_1 - \alpha_1\beta_2\gamma_1)\omega_\tau & (1+\alpha_2\beta_1\gamma_1 - \alpha_1\gamma_1)\omega_g & \beta_1\omega_y \\ (\gamma_1 + \beta_2\gamma_2)\omega_\tau & (\alpha_2\gamma_1 + \gamma_2)\omega_g & \omega_y \end{pmatrix} \begin{pmatrix} \epsilon_{\tau,t} \\ \epsilon_{g,t} \\ \epsilon_{y,t} \end{pmatrix}, \tag{31}$$

where $\Delta = (1 - \alpha_1\gamma_1 - \beta_1\gamma_2)$. Here, the element θ_{ij} of the marix (26) corresponds to the (i,j) element of the matrix in (31) divided by Δ, whereas $\epsilon_{1,t} = \epsilon_{\tau,t}$, $\epsilon_{2,t} = \epsilon_{g,t}$, and $\epsilon_{3,t} = \epsilon_{y,t}$.[31]

Blanchard and Perotti (2002) elaborate a set of identifying restrictions. This set fixes $\alpha_2 = 0$ such that taxes do not vary following a spending shock. It also calibrates $\alpha_1 = 2.08$ and $\beta_1 = 0$ using institutional information about tax and transfer systems, where such information allows to measure automatic adjustments of taxes and public spending rather than the government's systematic responses to fluctuations in output (see Blanchard and Perotti, 2002). Note that the three restrictions, implying that $\theta_{12} = \alpha_1\theta_{32}$, $\theta_{13} = \alpha_1\theta_{33}$, and $\theta_{23} = 0$, are placed on the subsystem relating the reduced-form innovations to the structural shocks $\epsilon_{g,t}$ and $\epsilon_{y,t}$, so that more restrictions are imposed than to fulfill the sufficient condtion.[32]

[31] Recall that the identified subsystem implies that the structural shock $\epsilon_{1,t}$ corresponds to the tax shock $\epsilon_{\tau,t}$. Also, the shocks $\epsilon_{2,t}$ and $\epsilon_{3,t}$ are ordered such that they can be interpreted as a spending shock $\epsilon_{g,t}$ and a non-fiscal shock $\epsilon_{y,t}$.

[32] An alternative set of identifying restrictions invoked by Blanchard and Perotti (2002) imposes $\beta_2 = 0$, $\alpha_1 = 2.08$, and $\beta_1 = 0$. These restrictions, implying that $\theta_{13} = \alpha_1\theta_{33}$ and $\theta_{23} = 0$, lead to the over-identification of the subsystem allowing to trace the responses of the variables to a spending shock.

Empirically, we place only one of the restrictions $\theta_{12} = \alpha_1\theta_{32}$, $\theta_{13} = \alpha_1\theta_{33}$, or $\theta_{23} = 0$ at a time, so that the subsystem linking the reduced-form innovations to the structural shocks $\epsilon_{g,t}$ and $\epsilon_{y,t}$ fulfilled the sufficient condition. The first restriction $\theta_{12} = \alpha_1\theta_{32}$ corresponds to the restriction that taxes do not contemporaneously respond to a government spending shock ($\alpha_2 = 0$). The second restriction $\theta_{13} = \alpha_1\theta_{33}$ comes from the constraint $\alpha_1 = 2.08$ calibrated by Blanchard and Perrotti (2002) and finally the last restriction $\theta_{23} = 0$ imposes that the automatic and government's systematic responses of government spending to changes in output is zero ($\beta_1 = 0$). The first and the third restrictions allow to obtain an estimator of the automatic and government's systematic responses of taxes to changes in output (α_1). Under the selected identifying restriction, we estimate the structural parameters α_i, β_i, γ_i, ω_j, and $\kappa^e_{\epsilon,11,11}$ (for $i = 1, 2$ and $j = \tau$, g, y) by minimizing the function (27).[33] Interestingly, the estimators of the parameter α_1 measuring the automatic and government's systematic responses of taxes to changes in output is close and not significantly different to the value calibrated by Blanchard and Perotti (2002) but seems at odds with high values obtained by Mertens and Ravn (2014) and Mountford and Uhlig (2009). To ease comparisons, Table 3 presents the resulting estimates of the elements θ_{ij} associated with system (31) and the fiscal multipliers. As expected, the estimates of the parameters θ_{11}, θ_{21}, θ_{31}, and $\kappa^e_{\epsilon,11,11}$, and the tax multiplier are virtually identical to those obtained from system (26). For the other parameters, some estimates differ substantially across the various identifying restrictions. This translates into a dynamic response of output after a government shock (see Figure 3) and a spending multiplier (i.e. the dollar change in output occurring in quarter $t + i$ resulting from a dollar increase in the exogenous component of government spending) that highly depends on the nature of the restriction: it is between 0.22 and 1.64 at impact, and it reaches a peak that ranges between 0.22 and 2.34 (see Table 3). This suggests that the evaluation of the effectiveness of the spending policy represents a challenging task.

6. Conclusion

In this paper, we first derived the sufficient condition for local, statistical identification of SVAR processes through higher unconditional moments. The condition is solely related to the numbers of structural shocks that display skewness and/or excess kurtosis. Furthermore, the condition establish which structural parameters are identified and which are not. For practitioners, this yields useful guidances about which structural parameters need to be restricted to achieve the identification of the entire system.

[33] These estimates are reported in Appendix F.

We then developed a tractable procedure to verify whether a SVAR process is identified, prior to the estimation of the structural parameters. In particular, the numbers of structural shocks exhibiting skewness and excess kurtosis correspond to the ranks of the third and fourth unconditional moment matrices of the reduced-form innovations. A bootstrap procedure is designed to improve the small-sample properties of these rank tests. The bootstrap version of the tests are virtually free of size distortions, whereas existing tests with asymptotic distributions suffer from severe size distortions even for large samples.

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Table 1. Empirical Sizes of Rank Tests: Kurtosis

	Asymptotic Distributions						Finite-Sample Distributions					
	$r^* = 0$						$r^* = 0$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	0.70	0.10	0.00	0.00	0.00	0.00	10.12	5.00	0.99	10.43	4.93	1.09
200	0.00	0.00	0.00	0.00	0.00	0.00	9.74	5.14	1.23	9.75	5.19	1.21
500	0.00	0.00	0.00	0.00	0.00	0.00	9.81	4.91	1.01	9.86	4.87	1.00
$1,000$	0.00	0.00	0.00	0.00	0.00	0.00	9.71	4.60	1.04	9.75	4.58	1.03
$5,000$	0.00	0.00	0.00	0.00	0.00	0.00	9.84	4.88	1.02	9.83	4.89	1.03
	$r^* = 1$						$r^* = 1$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	1.19	0.53	0.08	0.37	0.07	0.00	9.90	4.98	0.93	9.90	4.98	0.93
200	1.05	0.38	0.04	0.33	0.06	0.00	10.65	5.67	1.19	10.65	5.67	1.19
500	0.68	0.21	0.02	0.36	0.12	0.00	9.88	5.08	1.15	9.88	5.08	1.15
$1,000$	0.54	0.21	0.05	0.32	0.09	0.00	10.10	4.95	0.95	10.10	4.95	0.95
$5,000$	0.36	0.10	0.02	0.32	0.08	0.02	9.71	4.76	0.95	9.71	4.76	0.95

Notes. Entries are the empirical sizes (in percentage) of the rank tests with asymptotic and finite-sample distributions under the null hypothesis that $rk[K_u^e] = r^*$. The empirical sizes are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_1 = -0.5$, $\alpha_2 = 0.5$ and $\omega_1 = \omega_2 = 1$. Also, the distributions are $\epsilon_{2,t} \sim N(0,1)$, and i) $\epsilon_{1,t} \sim N(0,1)$ under $r^* = 0$ or ii) $1.291 \times \epsilon_{1,t} \sim t(5)$ under $r^* = 1$. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}^W_{r^*}$ and the likelihood-ratio (LR) statistic $\widehat{CRT}^{LR}_{r^*}$ associated with K_u^e exceed the critical values. The asymptotic critical values are computed as shown in Appendix D. The finite-sample critical values are computed by the bootstrap procedure elaborated in Section 4.2.

Table 2. Empirical Powers of Rank Tests with Finite-Sample Distributions: Kurtosis

	Excess Kurtosis = 1						*Excess Kurtosis* = 6					
	$r^* = 0$						$r^* = 0$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	32.46	22.77	10.40	32.00	22.41	10.23	64.00	54.52	37.60	63.17	53.69	36.42
200	45.88	36.18	19.45	45.39	35.64	18.74	85.58	80.00	65.25	84.96	79.23	64.38
500	73.88	65.63	46.76	73.86	65.06	45.81	99.28	98.73	96.46	99.22	98.66	96.06
1,000	93.20	89.67	78.58	93.20	89.41	78.41	100.0	100.0	100.0	100.0	100.0	100.0
	$r^* = 1$						$r^* = 1$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	18.96	12.57	3.90	18.96	12.57	3.90	53.18	44.34	23.81	53.18	44.34	23.81
200	32.72	24.31	10.27	32.72	24.31	10.27	80.78	74.44	55.78	80.78	74.44	55.78
500	99.09	98.58	95.55	99.09	98.58	95.55	99.09	98.58	95.55	99.09	98.58	95.55
1,000	100.0	99.99	99.94	100.0	99.99	99.94	100.0	99.99	99.94	100.0	99.99	99.94

Notes. Entries are the empirical powers (in percentage) of the rank tests with finite-sample distributions under the null hypothesis that $rk[K_u^e] = r^*$. The empirical powers are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_1 = -0.5$, $\alpha_2 = 0.5$ and $\omega_1 = \omega_2 = 1$. For $r^* = 0$, the distributions are: i) $\epsilon_{2,t} \sim N(0,1)$ and $1.118 \times \epsilon_{1,t} \sim t(10)$ when $\epsilon_{1,t}$ exhibits an excess kurtosis of 1, and ii) $\epsilon_{2,t} \sim N(0,1)$ and $1.291 \times \epsilon_{1,t} \sim t(5)$ when $\epsilon_{1,t}$ exhibits an excess kurtosis of 6. For $r^* = 1$, the distributions are: i) $1.118 \times \epsilon_{2,t} \sim t(10)$ and $1.118 \times \epsilon_{1,t} \sim t(10)$ when each shock exhibits an excess kurtosis of 1, and ii) $1.291 \times \epsilon_{2,t} \sim t(5)$ and $1.291 \times \epsilon_{1,t} \sim t(5)$ when each shock exhibits an excess kurtosis of 6. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}_{r^*}^{W}$ and the likelihood-ratio (LR) statistic $\widehat{CRT}_{r^*}^{LR}$ associated with K_u^e exceed the finite-sample critical values, where the latters are computed by the bootstrap procedure elaborated in Section 4.2.

Table 3. Parameter Estimates and Multipliers

	System (26)		System (31)		
			Parameter Estimates		
Parameter	$\theta_{23}=0$	$\theta_{32}=0$	$\theta_{12}=\alpha_1\theta_{32}$	$\theta_{13}=\alpha_1\theta_{33}$	$\theta_{23}=0$
θ_{11}	0.0474^{***}	0.0474^{***}	0.0473^{***}	0.0474^{***}	0.0474^{***}
θ_{12}	0.0026	-0.0005	$0.0006^{\dagger}$	0.0005	0.0036
θ_{13}	0.0089^{**}	0.0092^{**}	0.0098^{***}	$0.0095^{\dagger}$	0.0088^{**}
θ_{21}	0.0001	0.0001	-0.0001	0.0004	-0.0001
θ_{22}	0.0068^{***}	0.0064^{***}	0.0065^{***}	0.0067^{***}	0.0068^{***}
θ_{23}	$0.0000^{\dagger}$	0.0023^{**}	0.0019^{**}	-0.0007^{*}	$0.0000^{\dagger}$
θ_{31}	-0.0001	-0.0001	-0.0001	0.0001	-0.0001
θ_{32}	0.0017^{***}	$0.0000^{\dagger}$	0.0003	0.0022^{***}	0.0017^{***}
θ_{33}	0.0048^{***}	0.0051^{***}	0.0051^{***}	0.0046^{***}	0.0048^{***}
$\kappa^{e}_{\epsilon,11,11}$	2.7995^{***}	2.7867^{***}	2.8284^{***}	2.8135^{***}	2.8114^{***}
			Tax Multiplier		
Quarter	$\theta_{23}=0$	$\theta_{32}=0$	$\theta_{12}=\alpha_1\theta_{32}$	$\theta_{13}=\alpha_1\theta_{33}$	$\theta_{23}=0$
1	0.00	0.00	0.01	-0.01	0.00
4	0.05	0.05	0.07	0.06	0.06
8	0.24	0.24	0.26	0.25	0.24
Peak	0.61	0.61	0.62	0.63	0.60
	[14]	[14]	[14]	[14]	[14]
			Spending Multiplier		
Quarter			$\theta_{12}=\alpha_1\theta_{32}$	$\theta_{13}=\alpha_1\theta_{33}$	$\theta_{23}=0$
1			0.22	1.64^{***}	1.28^{***}
4			-0.71	1.93^{*}	1.24
8			-0.93	1.35	0.70
Peak			0.22	2.34^{***}	1.76^{***}
			[1]	[3]	[3]

Notes. Entries correspond to the estimates of the parameters of systems (26) and (31), and to the tax and spending multipliers. The tax (spending) multiplier measures the dollar change in output at a given horizon that results from a dollar decrease (increase) in the exogenous component of taxes (government spending). $*$, $**$, and $***$ indicate, respectively, that the 90, 95, and 99 percent confidence interval does not include zero, where the confidence intervals are computed from 5,000 bootstrap samples. $\dagger$ indicates that the parameter is constrained. Numbers between brackets indicate the quarters in which the maximum value of the multiplier is reached.

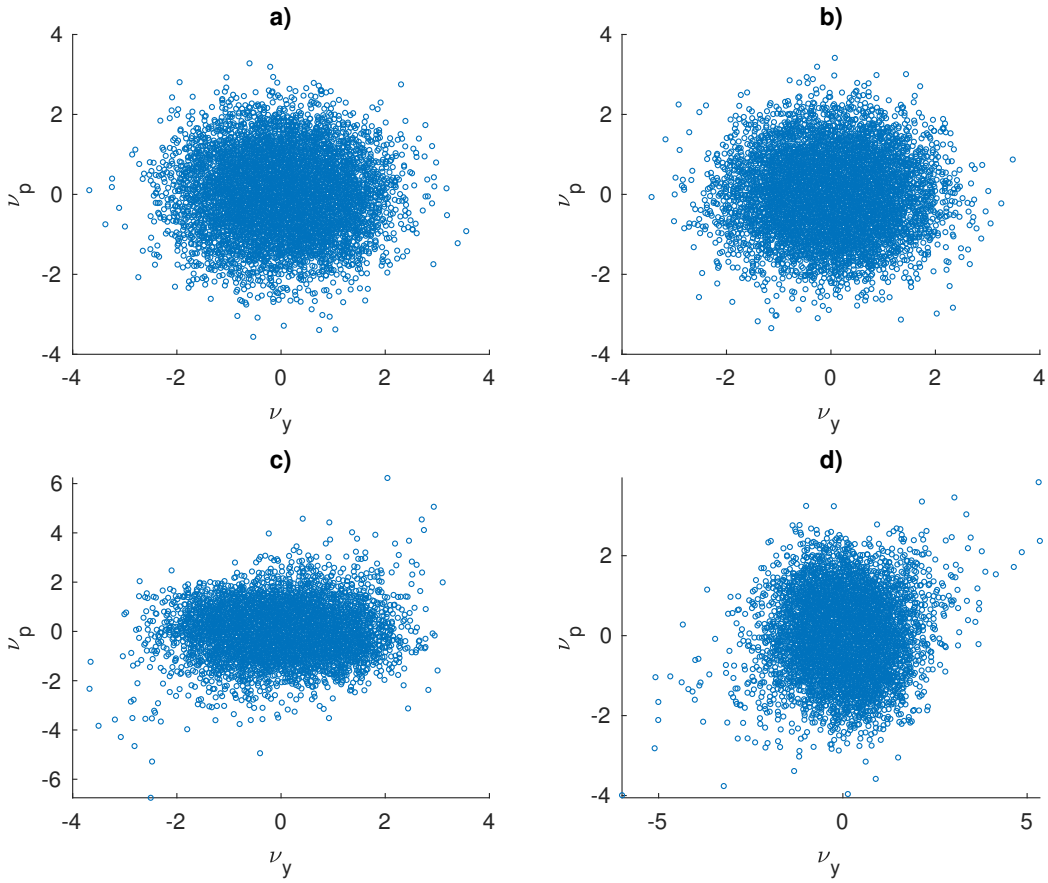

Figure 1a) contains the scatter plot of a simulated series with the following parametrization of equations (1) and (2) : $\alpha_d = -0.5$, $\alpha_s = 0.5$, $\omega_d = \omega_s = 1$, $\epsilon_{d,t} \sim N(0,1)$ and $\epsilon_{s,t} \sim N(0,1)$. Figure 1b) is for the same parametrization but when the impact matrix is multiplied by the orthogonal matrix Q defined in Section 2. Figure 1c) represents the scatter plot with the same parametrization than Figure 1a) but for $1.291 \times \epsilon_{d,t} \sim t(5)$ generating an excess kurtosis equals to $E[\epsilon_{d,t}^4] - 3 = 6$. Figure 1d) is for the same parametrization than Figure 1c) but when the impact matrix is multiplied by the orthogonal matrix Q defined in Section 2. The simulated series are generated from 10,000 draws.

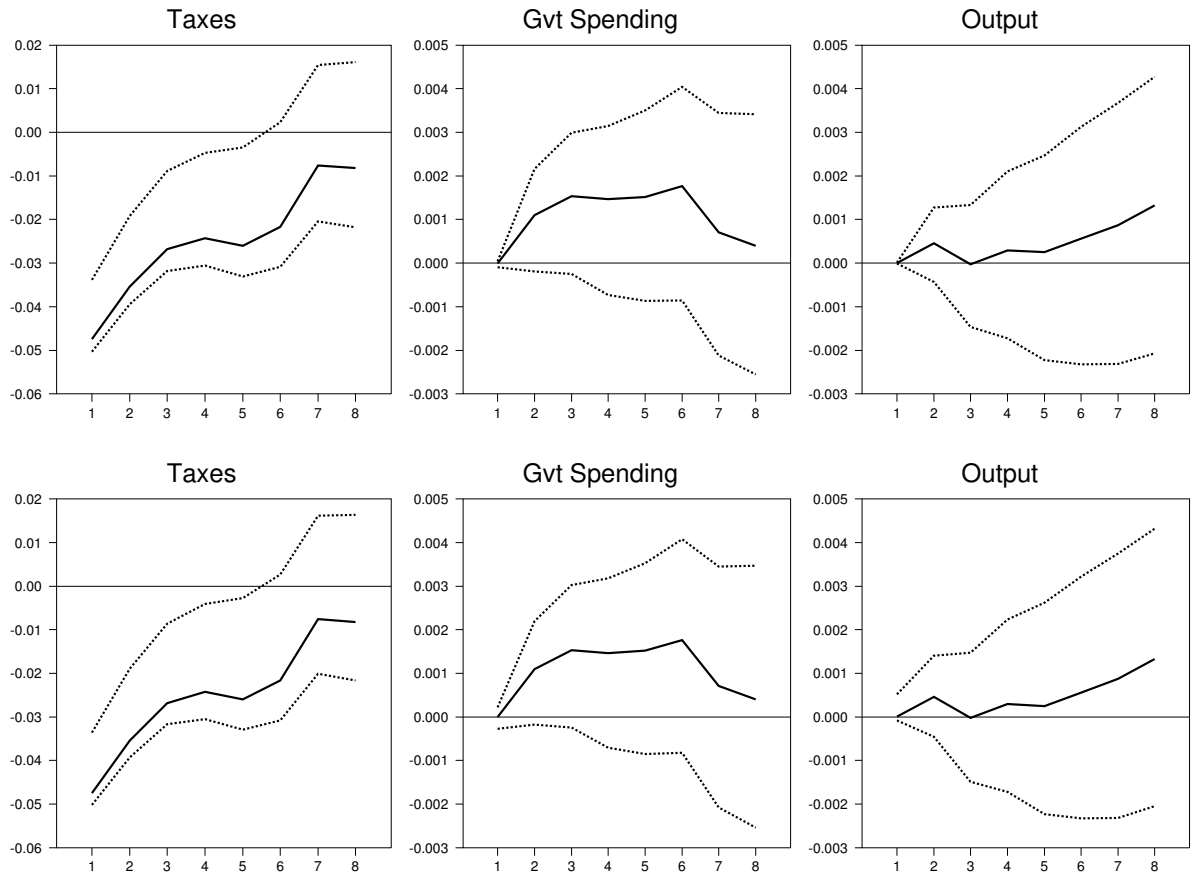

Figure 2. Dynamic responses following a tax shock. The solid lines are the dynamic responses of taxes, government spending, and output to a negative, one standard-deviation, tax shock, which are computed under the restriction $\theta_{23} = 0$ (first panel) and $\theta_{32} = 0$ (second panel). The dashed lines are the 95 % confidence intervals, which are computed from 5000 bootstrap samples.

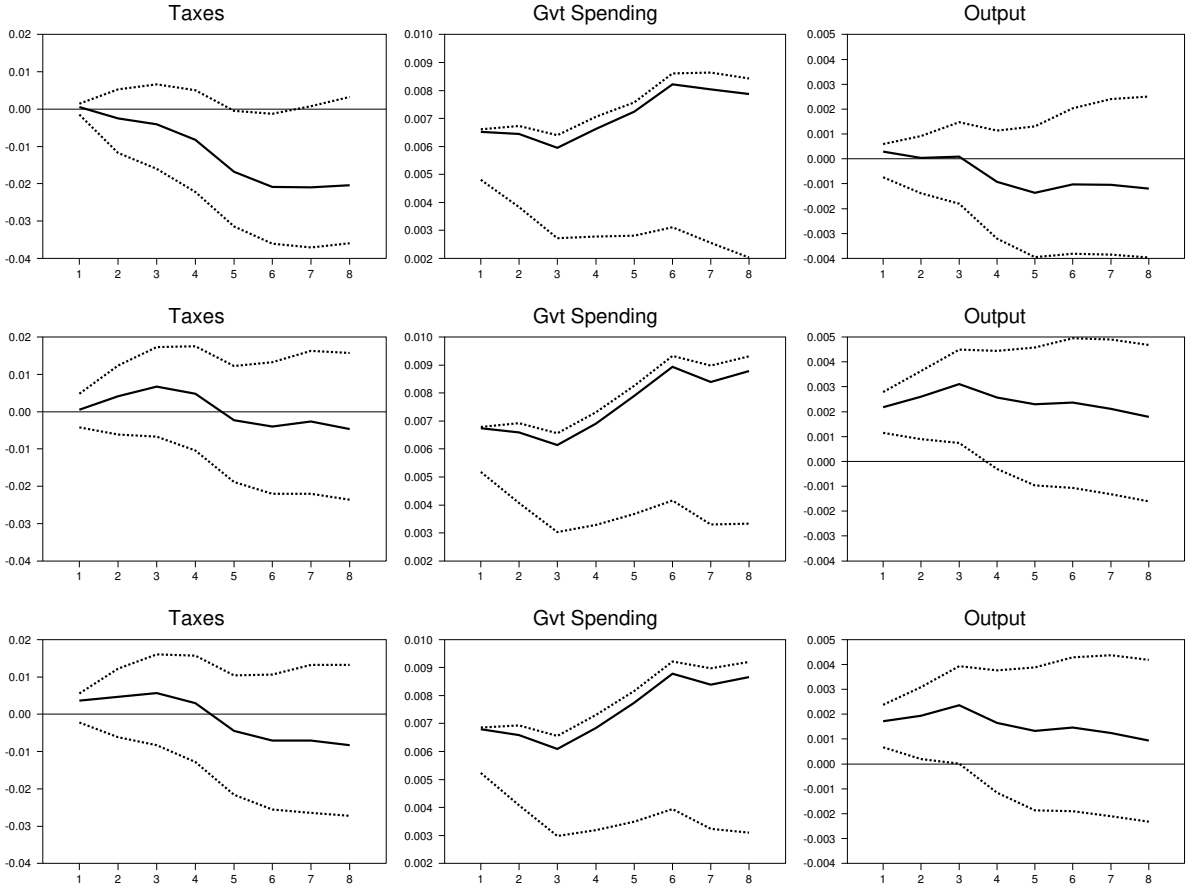

Figure 3. Dynamic responses following a government spending shock. The solid lines are the dynamic responses of taxes, government spending, and output to a positive, one standard-deviation, government spending shock, which are computed under the restriction $\theta_{12} = \alpha_1\theta_{32}$ (first panel), $\theta_{13} = \alpha_1\theta_{33}$ (second panel), and $\theta_{23} = 0$ (third panel). The dashed lines are the 95 % confidence intervals, which are computed from 5000 bootstrap samples.

Supplemental Material

Appendix A presents the rank condition for the local, statistical identification of SVAR processes with asymmetric structural shocks. Appendix B details the analytical partial derivatives involved in the Jacobian matrices related to the rank condition. Appendix C derives the rank condition. Appendix D contains the derivation of the asymptotic distribution of the rank test and a justification of the bootstrap procedure when $r^* > 0$. Appendix E documents the empirical sizes and powers of rank tests for symmetry. Appendix F reports the estimates of the structural parameters involved in system (30).

Appendix A: Identication under asymmetric structural shocks

The appendix elaborates the rank condition for a case which exploits only the skewness of the structural shocks. For this case, the relation between the reduced-form innovations and the structural shocks is partitioned as:

$$\nu_t = \begin{pmatrix} \Theta_s & \Theta_{ns} \end{pmatrix} \begin{pmatrix} \epsilon_{s,t} \\ \epsilon_{ns,t} \end{pmatrix}, \tag{A.1}$$

where $\epsilon_{s,t}$ and $\epsilon_{ns,t}$ contain the m_s and $(n - m_s)$ asymmetric and symmetric structural shocks.

Here, the number of parameters to identify is $\eta = n^2 + m_s$ because there are n^2 and m_s parameters to identify in Θ and S_ϵ. From the reduced form, $\rho = \left[\frac{n(n+1)}{2}\right] + \left[\frac{n(n+1)(n+2)}{6}\right]$ since there are $\frac{n(n+1)}{2}$ and $\frac{n(n+1)(n+2)}{6}$ distinct elements in Σ_ν and S_ν. The information contained in S_ν contributes to identify the parameters in Θ_s and S_ϵ, whereas Σ_ν contains specific information to identify the parameters in Θ_{ns}.

The sufficient rank condition holds when $r = \eta$. Under the short-run restrictions $R\theta_{ns} = q$, the rank condition is verified if:

$$rk[J^+] = rk\begin{bmatrix} J^+_{\theta_s} & J^+_{\theta_{ns}} & J^+_{s_\epsilon} \end{bmatrix} = rk\begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & J_{\sigma_\nu,s_\epsilon} \\ J_{s_\nu,\theta_s} & J_{s_\nu,\theta_{ns}} & J_{s_\nu,s_\epsilon} \\ 0 & R & 0 \end{bmatrix} = \eta, \tag{A.2}$$

where J^+ is the augmented Jacobian matrix, $J^+_{\theta_s} = \begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} & 0' \end{bmatrix}'$, $J^+_{\theta_{ns}} = \begin{bmatrix} J'_{\sigma_\nu,\theta_{ns}} & J'_{s_\nu,\theta_{ns}} & R' \end{bmatrix}'$, $J^+_{s_\epsilon} = \begin{bmatrix} J'_{\sigma_\nu,s_\epsilon} & J'_{s_\nu,s_\epsilon} & 0' \end{bmatrix}'$, and $J_{y,x} = \frac{\partial y}{\partial x'}$. Moreover, the vector σ_ν vectorizes the lower triangular part of the symmetric covariance matrix Σ_ν, and the vector s_ν collects the distinct elements of the coskewness matrix S_ν. Finally, the vector θ_s stacks the columns of the matrix Θ_s in system (A.1), the vector θ_{ns} contains the elements of the matrix Θ_{ns} and the vector s_ϵ includes the non-zero elements of the skewness matrix S_ϵ.

When no restrictions are placed on the structural parameters ($R = 0$), the rank of J is given by

$$rk[J] = rk\begin{bmatrix} J_{\theta_s} & J_{\theta_{ns}} & J_{s_\epsilon} \end{bmatrix} = rk\begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & J_{\sigma_\nu,s_\epsilon} \\ J_{s_\nu,\theta_s} & J_{s_\nu,\theta_{ns}} & J_{s_\nu,s_\epsilon} \end{bmatrix}$$

which is equal to $r = r_s + r_{ns} + r_{s_\epsilon}$ with $r_s = rk[J_{\theta_s}] = n \times m_s$, $r_{ns} = rk[J_{\theta_{ns}}] = \frac{n(n+1)}{2} - \frac{m_s(m_s+1)}{2}$, and $r_{s_\epsilon} = rk\left[J_{s_\epsilon}\right] = m_s$ as we show below. Consequently, the entire structural system is locally, statistically identified ($\eta = r$) when at least all, but one, structural shocks display non-zero skewnesses. Also, whether or not $\eta = r$, the parameters involve in Θ_s and S_ϵ are locally, statistically identified through the information contained in Σ_ν and S_ν. Hence, if the structural shocks of interest are asymmetric, then their effects are identified. When some restrictions are imposed on the structural parameters ($R \neq 0$), then the entire structural system is locally, statistically identified when $(\eta - r)$ linearly independent restrictions are imposed on the structural parameters contained in θ_{ns}. Thus, if the structural shocks of interest are symmetric, then their effects can only be determined when $(\eta - r)$ restrictions are placed on Θ_{ns}.

Appendix B: Analytical derivatives involved in the Jacobian matrices

This appendix presents the analytical partial derivatives involved in the Jacobian matrices for the cases (A.2), (17) and (18). First, the partial derivatives of the second unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$\begin{aligned} J_{\sigma_\nu,\theta_i} &= 2D_\sigma^+(\Theta \otimes I_n)\Upsilon_{\theta_i}, \\ J_{\sigma_\nu,s_\epsilon} &= 0, \\ J_{\sigma_\nu,\kappa_\epsilon^e} &= 0, \end{aligned}$$

where $i = s, ns$ in (A.1), $i = \kappa, n\kappa$ in (17), and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (18). The vectorization of the distinct elements of the second moments yields $\sigma_\nu = D_\sigma^+ vec(\Sigma_\nu)$, where $\sigma_\nu = vech(\Sigma_\nu)$, $D_\sigma^+ = (D_\sigma' D_\sigma)^{-1} D_\sigma'$, and D_σ is the $\left(n^2 \times \frac{n(n+1)}{2}\right)$ duplication matrix such that $D_\sigma \sigma_\nu = vec(\Sigma_\nu)$. Using this vectorization, we obtain $\frac{\partial \sigma_\nu}{\partial \theta_i'} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(\Theta)'} \frac{\partial vec(\Theta)}{\partial \theta_i'}$. Equation (11) leads to $vec(\Sigma_\nu) = (\Theta \otimes \Theta) vec(I_n)$, so that $\frac{\partial vec(\Sigma_\nu)}{\partial vec(\Theta)'} = 2(\Theta \otimes I_n)$ (see Lütkepohl, 2007, p. 363). Also, $\frac{\partial vec(\Theta)}{\partial \theta_i'} = \Upsilon_{\theta_i}$ is a matrix containing the values one and zero such that only the partial derivatives with respect to the elements of the vector θ_i are selected. As an example, consider the relation (A.1) with $n = 2$ and $m_s = 1$ (where the asymmetric structural shock is ordered first), then the $(n^2 \times nm_s)$ selection matrix corresponds to $\Upsilon_{\theta_s} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}'$ and $\theta_s = vec(\Theta_s)$. Moreover, $\frac{\partial \sigma_\nu}{\partial s_\epsilon'} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(S_\epsilon)'} \frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'}$, where $\frac{\partial vec(\Sigma_\nu)}{\partial vec(S_\epsilon)'} = 0$ given that Σ_ν is not a function of the skewnesses of the structural shocks. Likewise, $\frac{\partial \sigma_\nu}{\partial \kappa_\epsilon^{e\prime}} = D_\sigma^+ \frac{\partial vec(\Sigma_\nu)}{\partial vec(K_\epsilon^e)'} \frac{\partial vec(K_\epsilon^e)}{\partial \kappa_\epsilon^{e\prime}}$ with $\frac{\partial vec(\Sigma_\nu)}{\partial vec(K_\epsilon^e)'} = 0$.

Next, the partial derivatives of the third unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$\begin{aligned}
J_{s_\nu,\theta_i} &= D_s^+\{(I_{n^2}\otimes\Theta S_\epsilon)[(I_n\otimes C_{n,n}\otimes I_n)[(I_{n^2}\otimes vec(\Theta'))+(vec(\Theta')\otimes I_{n^2})]C_{n,n}] \\
&\quad +[(\Theta\otimes\Theta)S_\epsilon'\otimes I_n]\}\Upsilon_{\theta_i}, \\
J_{s_\nu,s_\epsilon} &= D_s^+(\Theta\otimes\Theta\otimes\Theta)\Upsilon_{s_\epsilon}, \\
J_{s_\nu,\kappa_\epsilon^e} &= 0,
\end{aligned}$$

where $i = s, ns$ in (A.1) and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (18). The vectorization of the distinct elements of the third moments corresponds to $s_\nu = D_s^+ vec(S_\nu)$, where $D_s^+ = (D_s'D_s)^{-1}D_s'$, and D_s is the $\left(n^3 \times \frac{n(n+1)(n+2)}{6}\right)$ matrix such that $D_s s_\nu = vec(S_\nu)$. As an example, for a bivariate system with $n = 2$, then:

$$D_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the above vectorization, we have $\frac{\partial s_\nu}{\partial\theta_i'} = D_s^+\frac{\partial vec(S_\nu)}{\partial vec(\Theta)'}\frac{\partial vec(\Theta)}{\partial\theta_i'}$ with $\frac{\partial vec(\Theta)}{\partial\theta_i'} = \Upsilon_{\theta_i}$. Rewriting equation (12) as $vec(S_\nu) = [(\Theta\otimes\Theta)\otimes\Theta]vec(S_\epsilon)$, then $\frac{\partial vec(S_\nu)}{\partial vec(\Theta)'} = (I_{n^2}\otimes\Theta S_\epsilon)\frac{\partial vec(\Theta'\otimes\Theta')}{\partial vec(\Theta)'} + [(\Theta\otimes\Theta)S_\epsilon'\otimes I_n]$, where $\frac{\partial vec(\Theta'\otimes\Theta')}{\partial vec(\Theta)'} = (I_n\otimes C_{n,n}\otimes I_n)[(I_{n^2}\otimes vec(\Theta'))+(vec(\Theta')\otimes I_{n^2})]\frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$ with $\frac{\partial vec(\Theta')}{\partial vec(\Theta)'} = C_{n,n}$ (see Magnus and Neudecker, 2007, pp. 208–209), and $C_{n,m}$ is a $(nm\times nm)$ commutation matrix implying that $C_{n,m}vec(A) = vec(A')$ for the arbitrary $(n\times m)$ matrix A. Note that $\frac{\partial s_\nu}{\partial\theta_i'} = 0$ for $i = ns$ in (A.2) and for $i = \kappa\kappa, ns\kappa$ in (18), since S_ν is not a function of the structural parameters relating the reduced-form innovations to the symmetric structural shocks. Furthermore, $\frac{\partial s_\nu}{\partial s_\epsilon} = D_s^+\frac{\partial vec(S_\nu)}{\partial vec(S_\epsilon)'}\frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'}$, where $\frac{\partial vec(S_\nu)}{\partial vec(S_\epsilon)'} = (\Theta\otimes\Theta\otimes\Theta)$ and $\frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'} = \Upsilon_{s_\epsilon}$ is a $(n^3\times m_s)$ matrix selecting the partial derivatives with respect to the non-zero elements of s_ϵ. In particular, for a system with $n = m_s = 2$, then Υ_{s_ϵ} has values one for the (1,1) and (8,2) elements, and zero elsewhere. For the system with $n = 2$ and $m_s = 1$, then Υ_{s_ϵ} has values one for the (1,1) element, and zero elsewhere. Moreover, $\frac{\partial s_\nu}{\partial\kappa_\epsilon^e} = D_s^+\frac{\partial vec(S_\nu)}{\partial vec(K_\epsilon^e)'}\frac{\partial vec(K_\epsilon^e)}{\partial\kappa_\epsilon^{e\prime}}$, where $\frac{\partial vec(S_\nu)}{\partial vec(K_\epsilon^e)'} = 0$ given that S_ν is not a function of the excess kurtoses of the structural shocks.

Let us now examine the rank of the matrices J_{σ_ν,θ_s}, $J_{s_\nu,\theta_{ns}}$ and J_{s_ν,s_ϵ}. As illustration, consider a system with $n = 2$,

$$\begin{pmatrix}\nu_{1,t}\\ \nu_{2,t}\end{pmatrix} = \begin{pmatrix}\theta_{11} & \theta_{12}\\ \theta_{21} & \theta_{22}\end{pmatrix}\begin{pmatrix}\epsilon_{1,t}\\ \epsilon_{2,t}\end{pmatrix} \tag{A.1}$$

For this example, the Jacobian matrix of the derivatives of the covariance matrix with respect to the parameters Θ is given by

$$J_{\sigma_\nu,\theta} = \begin{bmatrix} 2\theta_{11} & 0 & : & 2\theta_{12} & 0 \\ \theta_{21} & \theta_{11} & : & \theta_{22} & \theta_{12} \\ 0 & 2\theta_{21} & : & 0 & 2\theta_{22} \end{bmatrix}.$$

where $\theta = vec(\Theta)$. For a full rank matrix Θ, this matrix $J_{\sigma_\nu,\theta}$ is of rank $\frac{n(n+1)}{2}$ and each $\frac{n(n+1)}{2} \times n$ submatrix corresponding to the derivatives of $J_{\sigma_\nu,\theta}$ with respect to a column of the matrix Θ is of rank equals to n and this holds for $\forall\ n$. Also, the Jacobian matrix of the coskewness $J_{s_\nu,\theta}$ with respect to Θ for (A.1) is

$$J_{s_\nu,\theta} = \begin{bmatrix} 3\theta_{11}^2 s_{\epsilon,1,11} & 0 & : & 3\theta_{12}^2 s_{\epsilon,2,22} & 0 \\ 2\theta_{11}\theta_{21} s_{\epsilon,1,11} & \theta_{11}^2 s_{\epsilon,1,11} & : & 2\theta_{12}\theta_{22} s_{\epsilon,2,22} & \theta_{12}^2 s_{\epsilon,2,22} \\ \theta_{21}^2 s_{\epsilon,1,11} & 2\theta_{21}\theta_{11} s_{\epsilon,1,11} & : & \theta_{22}^2 s_{\epsilon,2,22} & 2\theta_{12}\theta_{22}^2 s_{\epsilon,2,22} \\ 0 & 3\theta_{21}^2 s_{\epsilon,1,11} & : & 0 & 3\theta_{22}^2 s_{\epsilon,2,22} \end{bmatrix}.$$

For a full rank matrix Θ, the Jacobian matrix $J_{s_\nu,\theta}$ of dimension $\frac{n(n+1)(n+2)}{6} \times n^2$ is of rank $n \times m_s$ which equals the rank of the matrix J_{s_ν,θ_s} since $J_{s_\nu,\theta_s} = J_{s_\nu,\theta}\Upsilon_{\theta_s}$. In the case above, for $m_s = 1$ (for instance when $s_{\epsilon,1,11} \neq 0$ and $s_{\epsilon,2,22} = 0$), the matrix J_{s_ν,θ_s} corresponds to the first two columns of $J_{s_\nu,\theta}$, whereas $J_{s_\nu,\theta_{ns}}$ corresponds to the two last columns of $J_{s_\nu,\theta}$. The rank of J_{s_ν,θ_s} and $J_{s_\nu,\theta}$ is equal to $n \times m_s = 2$. For $m_s = 2$, $J_{s_\nu,\theta_s} = J_{s_\nu,\theta}$ and the rank is $n \times m_s = 4$. For the general case, rearranging the rows of the matrix $J_{s_\nu,\theta}$ corresponding to the k-th column vector $\theta_{\bullet,k}$ of the matrix Θ, leads to the following $\frac{n(n+1)(n+2)}{6} \times n$ matrix

$$J^*_{s_\nu,\theta_{\bullet,k}} = \begin{bmatrix} B_{1k} \\ B_{2k} \\ \dots \\ B_{nk} \\ C_k \end{bmatrix} s_{\epsilon,k,kk}$$

where the matrix C_i is of dimension $\left(\frac{n(n+1)(n+2)}{6} - n^2\right) \times n$ for $n > 2$. The $n \times n$ matrices B_{lk} are given by $B_{lk} = \frac{\partial s_{\nu,l,l,j}}{\partial \theta'_{\bullet,k}}$ for $k, l, j = 1, \dots, n$ and C_k contains the derivatives of $s_{\nu,i,j,l}$ respective to $\theta'_{\bullet,k}$ for all $i < j < l$ for $i, j, l = 1, \dots, n$. Note that the column rank of $J_{s_\nu,\theta_{\bullet,k}}$ is the same as $J^*_{s_\nu,\theta_{\bullet,k}}$. Each matrix B_{lk} has the term $\theta_{lk}^2 s_{\epsilon,k,kk}$ on its diagonal except at the element l, k which is $3\theta_{lk}^2 s_{\epsilon,k,kk}$. The matrices B_{lk} are then of full column rank for all $\theta_{lk} \neq 0$. Given that Θ is of full rank, $J^*_{s_\nu,\theta_{\bullet,k}}$ (and then $J_{s_\nu,\theta_{\bullet,k}}$) is necessarily of full rank for $s_{\epsilon,k,kk} \neq 0$ and $J_{s_\nu,\theta_{\bullet,k}}$ cannot be collinear with $J_{s_\nu,\theta_{\bullet,k}}$ for $k \neq l$, $s_{\epsilon,k,kk} \neq 0$ and $s_{\epsilon,l,ll} \neq 0$. This shows that the Jacobian matrix $J_{s_\nu,\theta}$ is of rank equals to $n \times m_s$. For the illustration with $n = 2$, we get

$$J_{s_\nu,\theta} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

where B_{lk} are 2×2 matrices for $k, l = 1, 2$. For this case, $\frac{n(n+1)(n+2)}{6} - n^2 = 0$, so that there is no matrix C_k. We see that for each submatrix B_{lk}, the diagonal elements are function of $\theta_{lk}^2 s_{\epsilon,k,kk}$. For a full rank matrix Θ, the first two columns corresponding to B_{11} and B_{21} are of full rank (when $s_{\epsilon,1,11} \neq 0$) and they cannot be colinear with the last two columns corresponding to B_{12} and B_{22} (when $s_{\epsilon,2,22} \neq 0$).

For the Jacobian matrix J_{s_ν, s_ϵ}, the rank can be easily shown. The expression $(\Theta \otimes \Theta \otimes \Theta)$ is a square full rank matrix, so $(\Theta \otimes \Theta \otimes \Theta)\Upsilon_{s_\epsilon}$ is of the same column rank than Υ_{s_ϵ}, namely m_s. Since D_s^+ is a full column rank, $D_s^+(\Theta \otimes \Theta \otimes \Theta)\Upsilon_{s_\epsilon}$ has a rank equals to m_s.[34] For (A.1),

$$J_{s_\nu, s_\epsilon} = \begin{bmatrix} \theta_{11}^3 & : & \theta_{12}^3 \\ \theta_{11}^2\theta_{21} & : & \theta_{12}^2\theta_{22} \\ \theta_{11}\theta_{21}^2 & : & \theta_{12}\theta_{22}^2 \\ \theta_{21}^3 & : & \theta_{22}^3 \end{bmatrix} \Upsilon_{s_\epsilon}.$$

The rank of this matrix equals the rank of Υ_{s_ϵ} which equals m_s. However, the rank of $[J_{s_\nu,\theta} \;\; J_{s_\nu,s_\epsilon}]$ equals the rank of the matrix $J_{s_\nu,\theta}$ namely $n \times m_s$ given that $J_{s_\nu,\theta_{\bullet,k}} \times \theta_{\bullet,k} = 3 s_{\epsilon,k,kk} J_{s_\nu,s_\epsilon,k}$ where k indexes the column of the respective matrix. This holds for $\forall\, n$ for a full rank matrix Θ.

Finally, the partial derivatives of the fourth unconditional moments of the reduced-form innovations with respect to the structural parameters are:

$$\begin{aligned}
J_{\kappa_\nu^e,\theta_i} &= D_\kappa^+\{(I_{n^2} \otimes \Theta K_\epsilon^e)(I_{n^2} \otimes C_{n,n^2} \otimes I_n)[(I_{n^4} \otimes vec(\Theta'))(I_n \otimes C_{n,n} \otimes I_n) \times [(I_{n^2} \otimes vec(\Theta') \\
&\quad +(vec(\Theta') \otimes I_{n^2})]C_{n,n} + (vec(\Theta' \otimes \Theta') \otimes I_{n^2})C_{n,n}] + [(\Theta \otimes \Theta \otimes \Theta)K_\epsilon^{e\prime} \otimes I_n]\}\Upsilon_{\theta_i}, \\
J_{\kappa_\nu^e,s_\epsilon} &= 0, \\
J_{\kappa_\nu^e,\kappa_\epsilon^e} &= D_\kappa^+(\Theta \otimes \Theta \otimes \Theta \otimes \Theta)\Upsilon_{\kappa_\epsilon^e},
\end{aligned}$$

where $i = \kappa, n\kappa$ in (17) and $i = ss, \kappa\kappa, s\kappa, ns\kappa$ in (18). The vectorization of the distinct elements of the fourth moments is $\kappa_\nu^e = D_\kappa^+ vec(K_\nu^e)$, where $D_\kappa^+ = (D_\kappa' D_\kappa)^{-1} D_\kappa'$, and D_κ is the

[34] If A is a full column rank matrix and B is conformable for the multiplication AB, the $rk(AB) = rk(B)$.

$\left(n^4 \times \frac{n(n+1)(n+2)(n+3)}{24}\right)$ matrix such that $D_\kappa \kappa_\nu^e = vec(K_\nu^e)$. For example, when $n = 2$, then:

$$D_\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using the above vectorization, we have $\frac{\partial \kappa_\nu^e}{\partial \theta_i'} = D_\kappa^+ \frac{\partial vec(K_\nu^e)}{\partial vec(\Theta)'} \frac{\partial vec(\Theta)}{\partial \theta_i'}$ with $\frac{\partial vec(\Theta)}{\partial \theta_i'} = \Upsilon_{\theta_i}$. Given that equation (13) implies $vec(K_\nu^e) = [(\Theta \otimes \Theta \otimes \Theta) \otimes \Theta] vec(K_\epsilon^e)$, then $\frac{\partial vec(K_\nu^e)}{\partial vec(\Theta)'} = (I_{n^2} \otimes \Theta K_\epsilon^e) \frac{\partial vec(\Theta' \otimes \Theta' \otimes \Theta')}{\partial vec(\Theta)'} + [(\Theta' \otimes \Theta' \otimes \Theta') K_\epsilon^{e\prime} \otimes I_n]$, where $\frac{\partial vec(\Theta' \otimes \Theta' \otimes \Theta')}{\partial vec(\Theta)'} = (I_{n^2} \otimes C_{n,n^2} \otimes I_n) \left[(I_{n^4} \otimes vec(\Theta')) \frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} + [vec(\Theta' \otimes \Theta') \otimes I_{n^2}]\right] \frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$, and, as shown above, $\frac{\partial vec(\Theta' \otimes \Theta')}{\partial vec(\Theta)'} = (I_n \otimes C_{n,n} \otimes I_n)[(I_{n^2} \otimes vec(\Theta')) + (vec(\Theta') \otimes I_{n^2})] \frac{\partial vec(\Theta')}{\partial vec(\Theta)'}$ and $\frac{\partial vec(\Theta')}{\partial vec(\Theta)'} = C_{n,n}$. Note that $\frac{\partial \kappa_\nu^e}{\partial \theta_i'} = 0$ for $i = n\kappa$ in (18) and for $i = ss, ns\kappa$ in (19), since K_ν^e is not a function of the structural parameters relating the reduced-form innovations to the mesokurtic structural shocks. Moreover, $\frac{\partial \kappa_\nu^e}{\partial s_\epsilon} = D_\kappa^+ \frac{\partial vec(K_\nu^e)}{\partial vec(S_\epsilon)'} \frac{\partial vec(S_\epsilon)}{\partial s_\epsilon'}$, where $\frac{\partial vec(K_\nu^e)}{\partial vec(S_\epsilon)'} = 0$ given that K_ν^e is not a function of the skewnesses of the structural shocks. In addition, $\frac{\partial \kappa_\nu^e}{\partial \kappa_\epsilon^e} = D_\kappa^+ \frac{\partial vec(K_\nu^e)}{\partial vec(K_\epsilon^e)'} \frac{\partial vec(K_\epsilon^e)}{\partial \kappa_\epsilon^{e\prime}}$, where $\frac{\partial vec(K_\nu^e)}{\partial vec(K_\epsilon^e)'} = (\Theta \otimes \Theta \otimes \Theta \otimes \Theta)$ and $\frac{\partial vec(K_\epsilon^e)}{\partial \kappa_\epsilon^{e\prime}} = \Upsilon_{\kappa_\epsilon^e}$ is a $(n^4 \times m_\kappa)$ matrix selecting the partial derivatives with respect to the non-zero elements of κ_ϵ^e. For example, when $n = m_\kappa = 2$, then $\Upsilon_{\kappa_\epsilon^e}$ has values one for the (1,1) and (16,2) elements, and zero elsewhere. For the system with $n = 2$ and $m_\kappa = 1$, then $\Upsilon_{\kappa_\epsilon^e}$ has values one for the (1,1) element, and zero elsewhere.

Similarly to the case with skewed structural shocks, we can show that $rk[J_{\kappa_\nu^e,\theta}] = n \times m_\kappa$ and $rk[J_{\kappa_\nu^e,\kappa_\epsilon^e}] = m_\kappa$ for a full rank matrix Θ. In particular, the matrix $J_{\kappa_\nu^e,\theta_{\bullet,k}}$ has a form similar to the matrix $J_{s_\nu,\theta_{\bullet,k}}$ with elements function of θ_{lk}^3 on the diagonal of the block B_{lk}. Moreover, $rk\left[J_{\kappa_\nu^e,\theta}\ J_{\kappa_\nu^e,\kappa_\epsilon^e}\right] = n \times m_\kappa$ by noting that $J_{\kappa_\nu^e,\theta_{\bullet,k}} \times \theta_{\bullet,k} = 4\kappa_{\epsilon,kk,kk}^e J_{\kappa_\nu^e,\kappa_\epsilon^e,i}$ where k indexes the column of the respective matrix.

Appendix C: Rank condition

Let us now show that $rk[J] = r = r_s + r_{ns} + r_{s_\epsilon}$, as mentioned in appendix A. We need the following results for the rank of upper triangular block matrix :

Lemma 1 *Given that A is a $m \times n$ matrix, B is a $s \times t$ matrix and C is a $m \times t$ matrix,*

1.

$$rk(A) + rk(B) \leq rk\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) \leq rk(A) + rk\left(\begin{bmatrix} C \\ B \end{bmatrix}\right),$$

2.

$$rk(A) + rk(B) \leq rk\left(\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}\right) \leq rk\left(\begin{bmatrix} A & C \end{bmatrix}\right) + rk(B).$$

In Appendix B, it is shown that $rk\left[J_{s_\nu,\theta}\right] = rk\left[J_{s_\nu,\theta_s}\right] = n \times m_s$, $rk\left[J_{s_\nu,s_\epsilon}\right] = m_s$ and $rk\left[J_{s_\nu,\theta} \;\; J_{s_\nu,s_\epsilon}\right] = n \times m_s$. Moreover, each $\frac{n(n+1)}{2} \times n$ submatrix of $J_{\sigma_\nu,\theta}$ corresponding to each column of the matrix Θ is of rank equals to n. Now, we need to know the rank of the matrix of the derivative of the covariance matrix with respect to the parameters of the impact matrix J_{σ_ν,θ_s} and $J_{\sigma_\nu,\theta_{ns}}$. The rank of the first submatrix $rk[J'_{\sigma_\nu,\theta_s}] = \frac{n(n+1)}{2} - \frac{(n-m_s)(n-m_s+1)}{2}$ and for the second submatrix, the rank is equal to $rk[J'_{\sigma_\nu,\theta_{ns}}] = \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2}$. To understand this result, consider that $m_s = 1$. In this case, the $n \times n$ symmetric covariance matrix of the n-variables resulting from the skewed structural shock is of rank equals to one. Since only one row (column) is linear independent of the others rows (columns), this symmetric covariance matrix contains only n independent elements. The $n \times n$ symmetric covariance matrix of the n-variables resulting from the other structural shocks contains $n - m_s = n - 1$ linear independent rows (columns) which implies that this matrix has $n(n+1)/2 - 1$ idependent elements. For instance, suppose that $n = 3$ and $m_s = 1$ (where $\epsilon_{1,t}$ is the skewed structural shock), we get the following relationship:

$$\Sigma_\nu^{m_s} = \begin{bmatrix} \sigma^1_{\nu,11} & \sigma^1_{\nu,12} & \sigma^1_{\nu,13} \\ \sigma^1_{\nu,12} & \sigma^1_{\nu,22} & \sigma^1_{\nu,23} \\ \sigma^1_{\nu,13} & \sigma^1_{\nu,23} & \sigma^1_{\nu,33}. \end{bmatrix} = \begin{bmatrix} \theta_{11}^2 & \theta_{11}\theta_{21} & \theta_{11}\theta_{31} \\ \theta_{21}\theta_{11} & \theta_{21}^2 & \theta_{21}\theta_{31} \\ \theta_{31}\theta_{11} & \theta_{31}\theta_{21} & \theta_{31}^2 \end{bmatrix} = \begin{bmatrix} \theta_{11} \\ \theta_{21} \\ \theta_{31} \end{bmatrix} \begin{bmatrix} \theta_{11} & \theta_{21} & \theta_{31} \end{bmatrix} E(\epsilon_{1t}^2).$$

The rank of this matrix is equal to one because there is only one source of randomness; the skewed structural shock $\epsilon_{1,t}$. Consequently, only one row is linear independent of the other ones. This row contains n linear independent elements namely $\frac{n(n+1)}{2} - \frac{(n-m_s)(n-m_s+1)}{2} = 6 - 3 = 3$. The elements of the two other rows are linear combinations of this row. The rank of the symmetric covariance matrix for the n-variables induced by the two other structural shocks, denoted $\Sigma_\nu^{n-m_s}$, is :

$$\Sigma_\nu^{n-m_s} = \begin{bmatrix} \sigma^2_{\nu,11} & \sigma^2_{\nu,12} & \sigma^2_{\nu,13} \\ \sigma^2_{\nu,12} & \sigma^2_{\nu,22} & \sigma^2_{\nu,23} \\ \sigma^2_{\nu,13} & \sigma^2_{\nu,23} & \sigma^2_{\nu,33} \end{bmatrix}.$$

Since the rank of this submatrix is equal to the number of non-skewed structural shocks, there are two linear independent rows which contain $\frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} = 6-1 = 5$ independent elements for any combination of two rows of the matrix $\Sigma_\nu^{n-m_s}$. In the case where $m_s = 2$, there are two linear independent rows for the matrix $\Sigma_\nu^{m_s}$ which implies $\frac{n(n+1)}{2} - \frac{(n-m_s)(n-m_s+1)}{2} = 6-1 = 5$ independent elements and the matrix $\Sigma_\nu^{n-m_s}$ contains $\frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} = 6-1 = 3$ independent elements. As a result, the rank of Jacobian matrix $J_{\theta_s} = \begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} \end{bmatrix}'$ equals $n \times m_s$ by using $rk[J_{s_\nu,\theta_s}] = n \times m_s$ and $rk[J_{s_\nu,\theta_s}] \geq rk[J_{\sigma_\nu,\theta_s}]$. Now the rank of the Jacobian matrix $J_{\theta_{ns}} = \begin{bmatrix} J'_{\sigma_\nu,\theta_{ns}} & J'_{s_\nu,\theta_{ns}} \end{bmatrix}'$ is equal to the rank of the Jacobian matrix $J_{\sigma_\nu,\theta_{ns}}$ which is $\frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2}$ since $J_{s_\nu,\theta_{ns}}$ is a matrix of zeros. Finally, the rank of the matrix $J_{s_\epsilon} = \begin{bmatrix} J'_{\sigma_\nu,s_\epsilon} & J'_{s_\nu,s_\epsilon} \end{bmatrix}'$ is equal to the rank of the matrix J_{s_ν,s_ϵ} because only the coskewness matrix gives information about the third moment of the structural shocks. The rank of J_{s_ϵ} is $rk\left(J_{s_\nu,s_\epsilon}\right) = m_s$. The rank of the complete matrix of the Jacobian J respective to the structural parameters :

$$J = \begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} & 0 \\ J_{s_\nu,\theta_s} & 0 & J_{s_\nu,s_\epsilon} \end{bmatrix} \tag{C.1}$$

can then be shown to be equal to $rk[J] = r = r_s + r_{ns} + r_{s_\epsilon}$, where $r_s = n \times m_s$, $r_{ns} = \frac{n(n+1)}{2} - \frac{m_s(m_s+1)}{2}$ and $r_{s_\epsilon} = m_s$. First, consider the rank of the following block diagonal submatrix

$$\begin{bmatrix} J_{\sigma_\nu,\theta_{ns}} & 0 \\ 0 & J_{s_\nu,s_\epsilon} \end{bmatrix}. \tag{C.2}$$

The rank of this submatrix equals the sum of the rank of the block diagonal submatrices, namely $rk(J_{\sigma_\nu,\theta_{ns}}) + rk(J_{s_\nu,s_\epsilon}) = \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} + m_s$.

Second, the rank of (C.1) equal the rank of (C.2) plus the rank of J_{θ_s} except if there exists at least one linear combination of the columns from the matrix J_{θ_s} which corresponds to a column of (C.2). In the following, it is shown that such linear combination does not exist for a full rank matrix Θ. We show that such linear combination does not exist in two steps : i) there is no linear combination of J_{θ_s} which yields a column of $J_{\theta_{ns}}$ and ii) there is no linear combination of J_{θ_s} which yields a column of J_{s_ϵ}. For i), consider the submatrix $[J_{\theta_s} \;\; J_{\theta_{ns}}]$ which is

$$J_\theta = \begin{bmatrix} J_{\sigma_\nu,\theta_s} & J_{\sigma_\nu,\theta_{ns}} \\ J_{s_\nu,\theta_s} & 0 \end{bmatrix}.$$

The rank of J_θ equal to the rank of the submatrix J_{s_ν,θ_s} plus the rank of the submatrix $J_{\sigma_\nu,\theta_{ns}}$. Thus $rk(J_\theta) = n \times m_s + \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2}$. Indeed, the rank of the bloc matrix J_θ is equal to the rank of the matrix $\begin{bmatrix} J'_{\sigma_\nu,\theta_s} & J'_{s_\nu,\theta_s} \end{bmatrix}'$ plus the rank of the matrix $J_{\sigma_\nu,\theta_{ns}}$ using the following inequalities for the rank of upper triangular block matrix (Lemma 1):

$$rk(J_{\sigma_\nu,\theta_{ns}}) + rk(J_{s_\nu,\theta_s}) \leq rk(J_\theta) \leq rk(J_{\sigma_\nu,\theta_{ns}}) + rk\left(\begin{bmatrix} J_{\sigma_\nu,\theta_s} \\ J_{s_\nu,\theta_s} \end{bmatrix}\right).$$

Here, we have

$$rk(J_{\sigma_\nu,\theta_{ns}}) + rk(J_{s_\nu,\theta_s}) = rk(J_{\sigma_\nu,\theta_{ns}}) + rk\left(\begin{bmatrix} J_{\sigma_\nu,\theta_s} \\ J_{s_\nu,\theta_s} \end{bmatrix}\right).$$

For ii), we show that there is no linear combination of J_{θ_s} that yields a column of J_{s_ϵ}. In the preceding section, we show that $rk\left[J_{s_\nu,\theta_s} \;\; J_{s_\nu,\theta_{ns}} \;\; J_{s_\nu,s_\epsilon}\right] = rk\left[J_{s_\nu,\theta_s}\right]$ which implies that it exists an appropriated matrix A of dimension $(n \cdot m_s) \times m_s$ such that $\left[J_{s_\nu,\theta_s}\right] A = J_{s_\nu,s_\epsilon}$ since the submatrix $J_{s_\nu,\theta_{ns}} = \mathbf{0}$ is a matrix of zeros. Define each column of the matrix A by A_i for $i = 1, \ldots, m_s$.[35] For a matrix Θ of full rank, all $\frac{n(n+1)}{2} \times n$ submatrices $\left[J_{\sigma_\nu,\theta_{i,s}}\right]$ are necessarily of full rank so there is no vector such as $\left[J_{\sigma_\nu,\theta_{i,s}}\right] A_i = 0$ for $\forall i$ where i indexes the elements of the vector θ_s corrresponding to the column i of the matrix Θ_s. This implies that the rank of the matrix J equals $n \times m_s + \frac{n(n+1)}{2} - \frac{(m_s)(m_s+1)}{2} + m_s$. Given that $\left[J_{\sigma_\nu,\theta_{i,s}}\right] A_i \neq 0$ for $i = 1, \ldots, m_s$ and that J_{s_ν,θ_s} is of full rank, there is no linear combination of the columns of the matrix J_{θ_s} that that corresponds to a column of the matrix (C.2) since the Jacobian matrix respective of the structural parameter J_θ is of full rank. This completes the proof.

The same results hold for the case which exploits only the fourth moments of the structural shocks by modifying properly the dimension of the matrices and the notation.

For the general case

$$J = \begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} & J_{\sigma_\nu,\theta_{ns\kappa}} & J_{\sigma_\nu,s_\epsilon} & J_{\sigma_\nu,\kappa^e_\epsilon} \\ J_{s_\nu,\theta_{ss}} & J_{s_\nu,\theta_{\kappa\kappa}} & J_{s_\nu,\theta_{s\kappa}} & J_{s_\nu,\theta_{ns\kappa}} & J_{s_\nu,s_\epsilon} & J_{s_\nu,\kappa^e_\epsilon} \\ J_{\kappa^e_\nu,\theta_{ss}} & J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} & J_{\kappa^e_\nu,\theta_{ns\kappa}} & J_{\kappa^e_\nu,s_\epsilon} & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}$$

which equals

$$J = \begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} & J_{\sigma_\nu,\theta_{ns\kappa}} & 0 & 0 \\ J_{s_\nu,\theta_{ss}} & 0 & J_{s_\nu,\theta_{s\kappa}} & 0 & J_{s_\nu,s_\epsilon} & 0 \\ 0 & J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} & 0 & 0 & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix} \tag{C.3}$$

First, consider the block diagonal submatrix containing the last subgroup of columns

$$\begin{bmatrix} J_{\sigma_\nu,\theta_{ns\kappa}} & 0 & 0 \\ 0 & J_{s_\nu,s_\epsilon} & 0 \\ 0 & 0 & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}. \tag{C.4}$$

The rank of this submatrix equals the sum of the rank of the block diagonal submatrices, $rk(J_{\sigma_\nu,\theta_{ns\kappa}}) + rk(J_{s_\nu,s_\epsilon}) + rk(J_{\kappa^e_\nu,\kappa^e_\epsilon}) = \frac{n(n+1)}{2} - \frac{(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})(m_{ss}+m_{\kappa\kappa}+m_{s\kappa}+1)}{2} + m_s + m_\kappa$.

By an argument similar to the one above, the rank of the submatrix

$$\begin{bmatrix} J_{\sigma_\nu,\theta_{ss}} & J_{\sigma_\nu,\theta_{\kappa\kappa}} & J_{\sigma_\nu,\theta_{s\kappa}} \\ J_{s_\nu,\theta_{ss}} & 0 & J_{s_\nu,\theta_{s\kappa}} \\ 0 & J_{\kappa^e_\nu,\theta_{\kappa\kappa}} & J_{\kappa^e_\nu,\theta_{s\kappa}} \end{bmatrix} \tag{C.5}$$

[35] From Appendix B, A_i corresponds to the column of matrix θ_s divided by 3 times the respective measure of skewness.

equals the sum of rank of the submatrix $\left[J'_{\sigma_\nu,\theta_{ss}} \;\; J'_{s_\nu,\theta_{ss}}\right]'$ and the rank of $\left[J_{\kappa^e_\nu,\theta_{\kappa\kappa}} \;\; J_{\kappa^e_\nu,\theta_{s\kappa}}\right]$, using Lemma 1 and the fact that $rk\left[J_{\theta_{\kappa\kappa}} \;\; J_{\theta_{s\kappa}}\right] = rk\left[J_{\kappa^e_\nu,\theta_{\kappa\kappa}} \;\; J_{\kappa^e_\nu,\theta_{s\kappa}}\right] = n \times m_{\kappa\kappa} + n \times m_{s\kappa}$. The rank of (C.5) is then $n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa}$. Now, one needs to show that the rank of the complete Jacobian matrix (C.3) is the sum of the rank of (C.4) and (C.5). First, the rank of the submatrix containing (C.5) and $\left[J'_{\sigma_\nu,\theta_{ns\kappa}} \;\; 0' \;\; 0'\right]'$ equals the rank of (C.5) plus the rank of J_{σ_ν} by the lower triangular block structure of this submatrix (by Lemma 1) which is $n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa} + \frac{n(n+1)}{2} - \frac{(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})(m_{ss}+m_{\kappa\kappa}+m_{s\kappa}+1)}{2}$. By a proof similar to the one to the case under asymmetry only, for a full rank matrix Θ, there is no linear combination of (C.5) that can yield a column of the last two submatrices of (C.4), i.e.

$$\begin{bmatrix} 0 & 0 \\ J_{s_\nu,s_\epsilon} & 0 \\ 0 & J_{\kappa^e_\nu,\kappa^e_\epsilon} \end{bmatrix}.$$

The rank of J is then equals to $rk[J_{\theta_{ss}}] + rk[J_{\theta_{\kappa\kappa}}] + rk[J_{\theta_{s\kappa}}] + rk[J_{\theta_{ns\kappa}}] + rk[J_{s_\nu}] + rk[J_{\kappa^e_\epsilon}] = n \times m_{ss} + n \times m_{\kappa\kappa} + n \times m_{s\kappa} + \left(\frac{n(n+1)}{2} - \frac{(m_{ss}+m_{\kappa\kappa}+m_{s\kappa})(m_{ss}+m_{\kappa\kappa}+m_{s\kappa}+1)}{2}\right) + m_s + m_k$.

Finally, Corollary 1 results from that there is no linear combination of (C.5) that can yield a column of the last two submatrices of (C.4)

Appendix D: Asymptotic Distribution of the Rank Test

First, we derive the asymptotic distribution of the statistics $\widehat{CRT}^{LR}_{r^*}$ and $\widehat{CRT}^{W}_{r^*}$. Under the assumption in section 3.1 for K^e_ϵ, $E[\|\epsilon_t\|^8] < \infty$ and the estimator $\widehat{K}^e_u$ is a root-T consistent for the $n \times n^3$ excess cokurtosis matrix K^e_u of the normalized reduced-form innovations. In this context, the asymptotic distribution of $\widehat{K}^e_u$ is

$$T^{1/2} vec(\widehat{K}^e_u - K^e_u) \xrightarrow{\mathcal{L}} N(0, \Gamma)$$

where Γ is finite.

Now, suppose that the matrix K^e_u is of rank $r^* \leq n$. The singular value decomposition of K^e_u gives $K^e_u = C\Lambda D'$ where Λ is a diagonal matrix with the singular values on the diagonal. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the singular values of the matrix Λ ordered in decreasing values. For a matrix K^e_u of rank equal to r^*, the first r^* singular values are different from zero and the last $n - r^*$ singular values are equal to zero. Thus

$$C' K^e_u D = \begin{bmatrix} C'_{r^*} K^e_u D_{r^*} & C'_{r^*} K^e_u D_{n^3-r^*} \\ C'_{n-r^*} K^e_u D_{r^*} & C'_{n-r^*} K^e_u D_{n^3-r^*} \end{bmatrix} = \Lambda.$$

The submatrix $C'_{n-r^*} K^e_u D_{n^3-r^*}$ corresponds to the null space of K^e_u which is the object of interest (see Al-Sadoon, 2017). We have

$$\sum_{i=r^*+1}^{n} \hat{\lambda}^2_i = \|vec(\widehat{C}'_{n-r^*} \widehat{K}^e_{u,c} \widehat{D}_{n^3-r*})\|^2 = \|vec(\widehat{C}_{n-r^*} \widehat{C}'_{n-r^*} \widehat{K}^{e,b}_{u,c} \widehat{D}_{n^3-r*} \widehat{D}'_{n^3-r*})\|^2$$

where $U_{n-r^*} = C_{n-r^*}C'_{n-r^*}$ and $V_{n^3-r^*} = D_{n^3-r^*}D'_{n^3-r^*}$ are the orthogonal projectors onto the space spanned by the left and the right null space singular vectors.[36]

The vectorization of this matrix yields

$$vec\left(\widehat{U}_{n-r^*}\widehat{K}_u^e\widehat{V}_{n^3-r^*}\right) = \left(\widehat{V}_{n^3-r^*} \otimes \widehat{U}_{n-r^*}\right) vec(\widehat{K}_u^e).$$

Since $T^{1/2}vec(\widehat{K}_u^e - K_u^e) \to N(0,\Gamma)$, the convergence in probability of the orthogonal projectors $\widehat{U}_{n-r^*} \xrightarrow{\mathcal{P}} U_{n-r^*}$ and $\widehat{V}_{n^3-r^*} \xrightarrow{\mathcal{P}} V_{n^3-r^*}$[37] and $\widehat{\Gamma} \xrightarrow{\mathcal{P}} \Gamma$, this implies that

$$T^{1/2}\left(\widehat{V}_{n^3-r^*} \otimes \widehat{U}_{n-r^*}\right)' vec(\widehat{K}_u^e - K_u^e) \xrightarrow{\mathcal{L}} N\left(0, \left(V_{n^3-r^*} \otimes U_{n-r^*}\right)\Gamma\left(V_{n^3-r^*}. \otimes U_{n-r^*}\right)\right)$$

Statistics $\widehat{CRT}_{r^*}^{LR}$ and $\widehat{CRT}_{r^*}^{W}$ converge asymptotically to

$$Tr(X_{r^*}X'_{r^*}) + o_p(1) = vec(X_{r^*})'vec(X_{r^*}) + o_p(1)$$

where $X_{r^*} = T^{1/2}\left(V'_{n^3-r^*} \otimes U'_{n-r^*}\right) vec(\widehat{K}_u^e - K_u^e)$. Both statistics have a limiting distribution given by $\sum_{i=1}^{t^*}\delta_i^{r^*}Z_i^2$ where $\delta_1^{r^*} \geq \ldots \geq \delta_{t^*}^{r^*}$ are the non-zero ordered eigenvalues of the matrix $\left(V_{n^3-r^*} \otimes U_{n-r^*}\right)\Gamma\left(V_{n^3-r^*} \otimes U_{n-r^*}\right)$ and $\{Z_i\}_{i=1}^{t^*}$ are independent $N(0,1)$ variates. The limiting distribution is then a weighted sum of t^* independent chi-squared variables with one degree of freedom and the weights are given by the non-zero eigenvalues $\delta_i^{r^*}$ for $i = 1,\ldots,t^*$. An estimator of the cumulative distribution function is obtained using the estimated counterparts of the matrices U_{n-r^*}, $V_{n^3-r^*}$ and Γ and the c.d.f. of the corresponding weighted sum of Z_i^2 for $i = 1,\ldots,t^*$ which can be easily evaluated by simulation.

Now we show that the subvector $u_{r^*,t}^b$ obtained by bootstrapping the vector $\omega_{r^*,t}^{b'} = \widehat{C}'_{r^*}\hat{u}_t$ for $b = 1,\ldots,B$ implies that $\hat{\lambda}_i^b \xrightarrow{\mathcal{P}} \hat{\lambda}_i$ where $\hat{\lambda}_i^b$ are the bootstrap estimators of the r^* largest singular values and $\hat{\lambda}_i$ are the sample estimators. Suppose a vector z with the following relation with a vector u:

$$z_t = C'u_t$$

where C is orthonormal. We have the following relation for the excess cokurtosis

$$K_z^e = C'K_u^e\left(C \otimes C \otimes C\right)$$

[36] Unlike to Robin and Smith (2000) and Bura and Yang (2011) but similarly to Portier and Delyon (2014), we consider orthogonal projection matrices U_{n-r^*} and $V_{n^3-r^*}$. The orthogonal projection matrices are invariant to the choice of a basis while the singular vectors in C_{n-r^*} and $D_{n^3-r^*}$ are uniquely defined only up to post-multiplication by an orthogonal matrix in a case of a multiplicity of singular values. Moreover, the orthogonal projection is continuous in the elements of the matrix, a necessary condition to guarantee the convergence in probability (see Dufour and Valéry, 2012).

[37] See Al-Sadoon, 2017, Theorem 1.

For the quadratic form of the excess cokurtosis

$$K_z^e K_z^{e\prime} \quad = \quad C' K_u^e \left(C \otimes C \otimes C\right) \left(C' \otimes C' \otimes C'\right) K_u^{e\prime} C = C' K_{u,c}^e K_u^{e\prime} C.$$

By the eigenvalue decomposition $K_{u,}^e K_u^{e\prime} = C\Lambda^2 C'$ which implies $K_z^e K_z^{e\prime} = \Xi = diag(\lambda_1^2, \ldots, \lambda_{r*}^2, 0, \ldots, 0)$ for a matrix K_u^e of rank $r*$ with the eigenvalues in descending order, where the eigenvalues are the square of the singular values λ_i. Thus, linear combinations of the normalized reduced-form innovations $\omega_{r^*} = \widehat{C}'_{r^*} \hat{u}_t$ capture the excess cokurtosis of the vector of the normalized reduced-form innovations where $\widehat{C}_r^*$ are the first r^* columns of $\widehat{C}$ corresponding to the singular values $\lambda_1, \ldots, \lambda_{r^*}$. The subvector $u_{r^*,t}^b$ is generated by bootstrapping the vector $\omega'_{r^*,t} = \widehat{C}'_{r^*} \hat{u}_t$ for $b = 1, \ldots, B$. Thus, for a consistent estimator of the excess cokurtosis $\widehat{K}^e_{u^b_{r^*}}$ of $u_{r^*,t}^b$ for $b = 1, \ldots, B$, a given matrix $\widehat{C}_{r^*}$ and by the continuity of the singular values, $\hat{\lambda}_i^b(\widehat{K}^e_{u^b_{r^*}} \widehat{K}^{e\prime}_{u^b_{r^*}}) \xrightarrow{\mathcal{P}} \hat{\lambda}_i(\widehat{C}'_{r^*} \widehat{K}_u^e \widehat{K}_u^{e\prime} \widehat{C}_{r^*})$ for $i = 1, \ldots, r^*$.

Appendix E: Empirical sizes and powers of rank tests for symmetry

This appendix reports the empirical sizes and powers of rank tests for symmetry. Table E.1 shows the empirical sizes. The Wald test with asymptotic distributions has empirical sizes that slightly deviate from the nominal ones, and the likelihood-ratio test with limiting distributions has empirical sizes that are substantially smaller than the nominal counterparts. In contrast, both the Wald and likelihood-ratio tests with finite-sample distributions feature empirical sizes that are almost identical to the nominal sizes, regardless of the number of observations in the sample.

Table E.2 displays the empirical powers. For the Wald and likelihood-ratio tests with finite-sample distributions, the powers substantially improve as the sample size increases and as the structural shocks become more skewed.

Table E.1. Empirical Sizes of Rank Tests: Skewness

	Asymptotic Distributions						Finite-Sample Distributions					
	$r^* = 0$						$r^* = 0$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	8.72	3.92	0.53	2.68	0.63	0.01	9.42	4.65	0.98	9.56	4.85	1.01
200	9.99	4.66	0.80	5.81	1.91	0.12	10.17	5.25	0.98	10.19	5.20	1.00
500	9.93	4.69	0.81	7.97	3.36	0.41	10.14	5.04	1.10	10.29	4.99	1.12
1,000	9.73	4.63	0.70	8.65	3.94	0.52	9.82	4.91	0.92	9.87	4.90	0.92
5,000	10.03	5.22	1.09	9.90	4.97	1.02	10.02	5.10	1.12	9.98	5.11	1.11
	$r^* = 1$						$r^* = 1$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	11.83	5.79	1.52	7.86	3.22	0.51	11.41	6.35	1.47	11.41	6.35	1.47
200	10.87	5.30	1.18	8.60	3.66	0.53	9.11	4.86	1.42	9.11	4.86	1.42
500	10.89	5.20	1.06	9.74	4.42	0.63	9.29	4.55	1.07	9.29	4.55	1.07
1,000	9.97	4.82	1.03	9.45	4.36	0.86	8.39	4.26	1.02	8.39	4.26	1.02
5,000	10.61	5.59	1.02	10.05	5.47	0.99	9.20	4.68	0.96	9.20	4.68	0.96

Notes. Entries are the empirical sizes (in percentage) of the rank tests with asymptotic and finite-sample distributions under the null hypothesis that $rk[S_u] = r^*$. The empirical sizes are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_1 = -0.5$, $\alpha_2 = 0.5$ and $\omega_1 = \omega_2 = 1$. Also, the distributions are $\epsilon_{2,t} \sim N(0,1)$, and i) $\epsilon_{1,t} \sim N(0,1)$ under $r^* = 0$ or ii) $2.1755 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{1,t} \sim N(-3.7326, 1)$ with probability 0.2113 under $r^* = 1$. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}^{W}_{r^*}$ and the likelihood-ratio (LR) statistic $\widehat{CRT}^{LR}_{r^*}$ associated with S_u exceed the critical values. The asymptotic critical values are computed as shown in Appendix D. The finite-sample critical values are computed by the bootstrap procedure elaborated in Section 4.2.

Table E.2. Empirical Powers of Rank Tests with Finite-Sample Distributions: Skewness

	$Skewness = -0.5231$						$Skewness = -0.9907$					
	$r^* = 0$						$r^* = 0$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	20.71	11.44	2.42	20.88	11.46	2.53	72.05	46.66	10.43	69.95	44.82	10.53
200	41.02	26.70	8.50	40.58	26.40	8.15	99.35	96.85	74.28	99.23	96.33	67.90
500	82.98	71.28	42.66	82.82	70.93	41.24	100.0	100.0	100.0	100.0	100.0	100.0
$1,000$	99.11	97.66	88.94	99.10	97.64	88.51	100.0	100.0	100.0	100.0	100.0	100.0
	$r^* = 1$						$r^* = 1$					
	Wald			LR			Wald			LR		
T	10 %	5%	1%	10 %	5%	1%	10%	5%	1%	10%	5%	1%
100	16.35	8.05	1.31	16.35	8.05	1.31	88.27	78.73	41.91	89.15	78.75	41.91
200	41.12	27.24	8.06	41.12	27.24	8.06	99.70	99.20	94.65	99.70	99.20	94.65
500	86.85	78.10	53.80	86.85	78.10	53.80	100.0	100.0	100.0	100.0	100.0	100.0
$1,000$	99.49	98.65	94.17	99.49	98.65	94.17	100.0	100.0	100.0	100.0	100.0	100.0

Notes. Entries are the empirical powers (in percentage) of the rank tests with finite-sample distributions under the null hypothesis that $rk[S_u] = r^*$. The empirical powers are evaluated for the bivariate specification (1)–(2), where the parameters are set as follows: $\alpha_1 = -0,5$, $\alpha_2 = 0.5$ and $\omega_1 = \omega_2 = 1$. For $r^* = 0$, the distributions are: i) $\epsilon_{2,t} \sim N(0,1)$ as well as $1.6808 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.5 and $1.6808 \times \epsilon_{1,t} \sim N(-1, 2.65)$ with probability 0.5 when $\epsilon_{1,t}$ exhibits a skewness of -0.5231, and ii) $\epsilon_{2,t} \sim N(0,1)$ as well as $2.1755 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.7887 and $2.1755 \times \epsilon_{1,t} \sim N(-3.7326, 1)$ with probability 0.2113 when $\epsilon_{1,t}$ exhibits a skewness of -0.9907. For $r^* = 1$, the distributions are: i) $1.6808 \times \epsilon_{2,t} \sim N(1,1)$ and $1.6808 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.5 as well as $1.6808 \times \epsilon_{2,t} \sim N(-1, 2.65)$ and $1.6808 \times \epsilon_{1,t} \sim N(-1, 2.65)$ with probability 0.5 when each shock exhibits a skewness of -0.5231, and ii) $2.1755 \times \epsilon_{2,t} \sim N(1,1)$ and $2.1755 \times \epsilon_{1,t} \sim N(1,1)$ with probability 0.7887 as well as $2.1755 \times \epsilon_{2,t} \sim N(-3.7326, 1)$ and $2.1755 \times \epsilon_{1,t} \sim N(-3.7326, 1)$ with probability 0.2113 when each shock exhibits a skewness of -0.9907. For each parametrization, 10,000 simulated samples of size T are generated to compute the proportions of time that the Wald statistic $\widehat{CRT}^{W}_{r^*}$ and the likelihood-ratio (LR) statistic $\widehat{CRT}^{LR}_{r^*}$ associated with S_u exceed the finite-sample critical values, where the latters are computed by the bootstrap procedure elaborated in Section 4.2.

Appendix F: Estimates of the structural parameters

Table F.1 shows the estimates of the structural parameters involved in system (30).

Table F.1. Parameter Estimates

Parameter	$\alpha_2 = 0$	$\alpha_1 = 2.08$	$\beta_1 = 0$
α_1	1.9409^{***}	$2.0800^{\dagger}$	1.8359^{**}
α_2	$0.0000^{\dagger}$	-0.5711^{*}	0.0728
β_1	0.3797^{**}	-0.1482^{*}	$0.0000^{\dagger}$
β_2	-0.0015	0.0095^{*}	-0.0030
γ_1	-0.0013	-0.0021	0.0002
γ_2	0.0439	0.3235^{***}	0.2516^{***}
ω_τ	0.0474^{***}	0.0473^{***}	0.0474^{***}
ω_g	0.0064^{***}	0.0071^{***}	0.0068^{***}
ω_y	0.0050^{***}	0.0048^{***}	0.0048^{***}
$\kappa^e_{\epsilon,11,11}$	2.8284^{***}	2.8135^{***}	2.8114^{***}

Notes. Entries correspond to the estimates of the parameters of system (30). $*$, $**$, and $***$ indicate, respectively, that the 90, 95, and 99 percent confidence interval does not include zero, where the confidence intervals are computed from 5,000 bootstrap samples. $\dagger$ indicates that the parameter is constrained. The restrictions $\alpha_2 = 0$, $\alpha_1 = 2.08$, and $\beta_1 = 0$ imply that $\theta_{12} = \alpha_1\theta_{32}$, $\theta_{13} = \alpha_1\theta_{33}$, and $\theta_{23} = 0$.